MICRO-428: Metrology

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MICRO-428: Metrology

Week Nine: Elements of Statistics

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Reference Books (Weeks 8&9)

- J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015
- A. Papoulis, Probability, *Random Variables and Stochastic Processes*, 3rd ed., 1991
- S.M. Ross, *Introduction to Probability Models*, 10th ed., 2009
- I.G. Hughes, T.P.A. Hase, *Measurements and their Uncertainties*, 1st ed., 2010
- G.E.P. Box, J.S. Hunter, W.G. Hunter, *Statistics for Experimenters*, 2nd ed., 2005
- J.R. Taylor, *An Introduction to Error Analysis*, 2nd ed., 1997

Week 8 Summary

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8.1 **Introduction to Probability**: $P\{\mathcal{A}\}, P\{\mathcal{A}|\mathcal{B}\} \rightarrow \text{Bayes' rule, Law of Total Prob. (LOTP)}$,

Independent Variables

8.2 **Random Variables**: discrete/continuous RV X and its distribution expressed as

PMF $p_X(x)$ / PDF $f_X(x) \leftrightarrow$ CDF $F_X(x)$

Examples: Binomial: $\text{Bin}(n, p)$, Poisson: $X \sim \text{Pois}(\lambda)$, Uniform: $U \sim \text{Unif}(a, b)$, Normal (Gaussian): $X \sim \mathcal{N}(\mu, \sigma^2)$, Exponential: $X \sim \text{Expo}(\lambda)$

8.3 **Moments**: RV X: expected value (mean) $E\{X\}$, variance $Var\{X\} = \sigma^2$ /standard deviation $SD\{X\} = \sqrt{Var\{X\}} = \sigma \rightarrow n$-th moment $E\{X^n\}$, central moment /standardized moment and their properties $\leftarrow$ moment generating function (MGF) $\phi(t) = E\{e^{tX}\}$

Week 8 Summary

S

8.4 **Covariance and Correlation**:

Multiple RVs → Multivariate distributions (8.1, 8.2 →): joint → marginal, → conditional, Independent distributions

Covariance $Cov\{X,Y\} \rightarrow \ Corr\{X,Y\}$ (unitless version)

Variance of multivariate distributions:

1. $Var\{X+Y\} = Var\{X\} + Var\{Y\} + 2Cov\{X,Y\}$

2. $Var\{X_1 + \cdots + X_n\} = Var\{X_1\} + \cdots + Var\{X_n\} + 2\sum_{i<j} Cov\{X_i, X_j\}$

Outline

9.0.1 Uniform Distribution

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- Uniform random variable in (a, b): completely random number between a and b

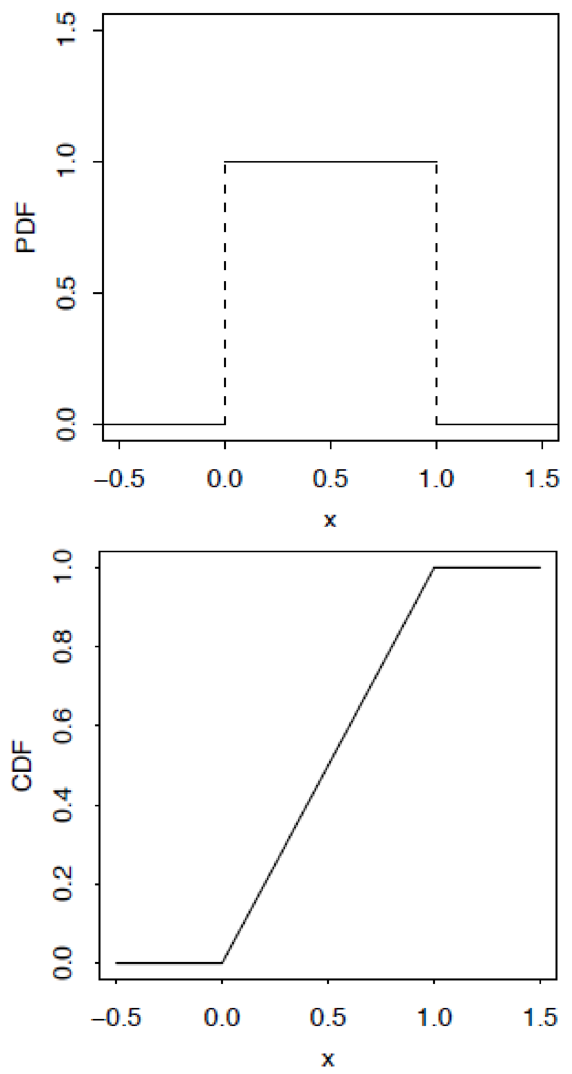
  -> PDF constant over chosen interval

- Uniform distribution $U \sim \mathrm{Unif}(a, b)$ in the interval (a, b) if:

$$\text{PDF:} \quad f_U(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{CDF:} \quad F_U(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \ge b \end{cases}$$

Unif(0,1) PDF & CDF



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.2

9.0.1 Uniform Distribution (contd.)

- Probability is inversely proportional to length.

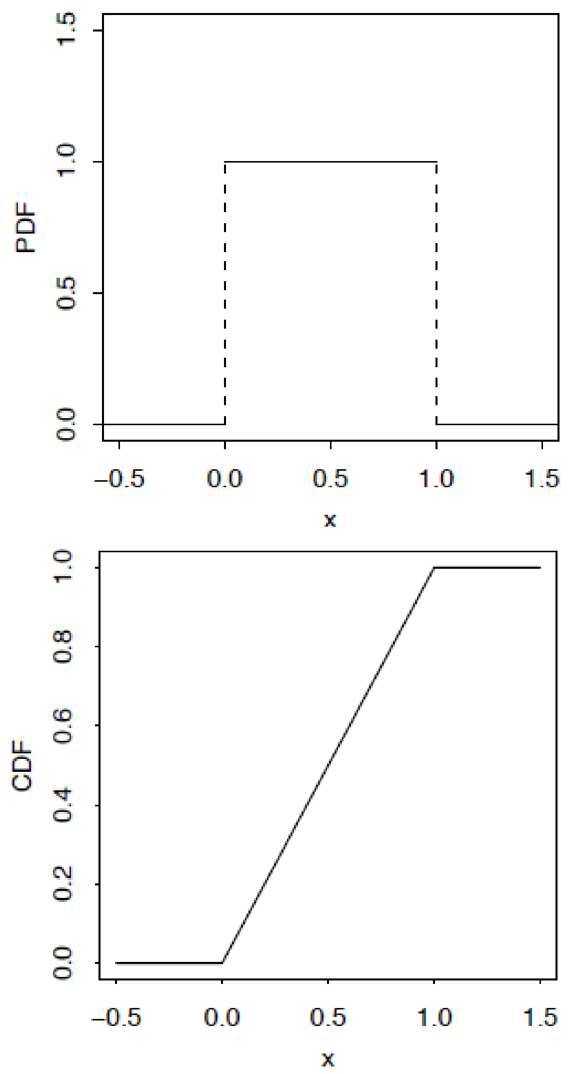
- Even in a sub-interval, we still have a uniform distribution

$$\text{Mean:} \quad E\{U\} = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$$

$$\text{Second Order Moment:} \quad E\{U^2\} = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{1}{3}\frac{b^3-a^3}{b-a}$$

$$\text{Variance}*: \quad Var\{U\} = E\{U^2\} - (E\{U\})^2 = \frac{1}{3}\frac{b^3-a^3}{b-a} - \left(\frac{a+b}{2}\right)^2 =$$
$$= \frac{(b-a)^2}{12}$$

Unif(0,1) PDF & CDF



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.2

**Using 8.3.4 (W8)*

9.0.2 Standard Gaussian Distribution

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- Gaussian (or Normal) distribution:
  - well-known continuous distribution with a bell-shaped PDF
  - widely used in statistics because of the central limit theorem (see next section)

- Standard Gaussian $Z \sim \mathcal{N}(0,1)$:

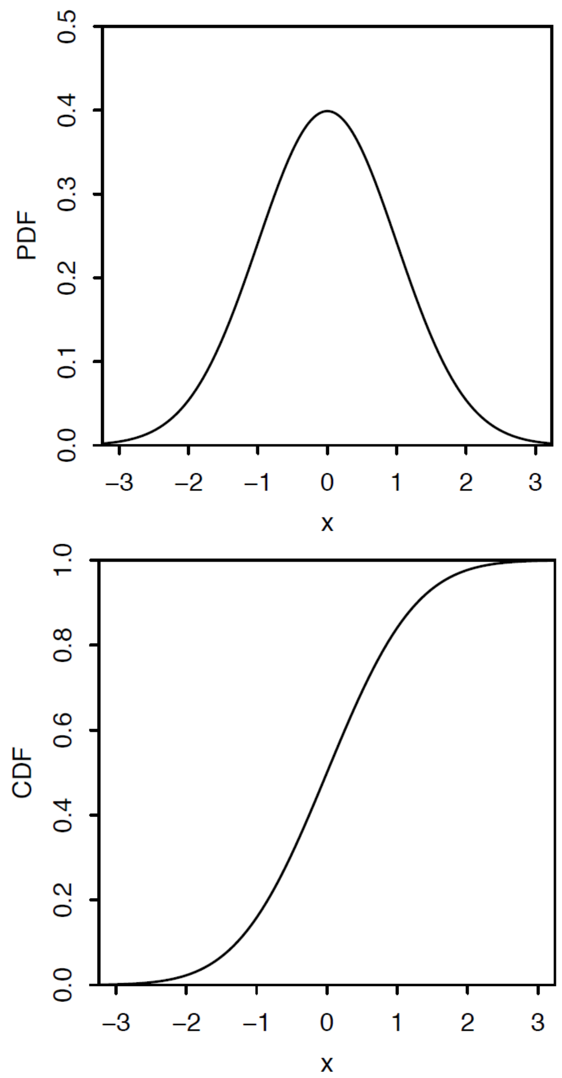
$$\text{PDF:} \quad \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty$$

$$\text{CDF:} \quad \Phi(z) = \int_{-\infty}^{z} \varphi(t)dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

No closed form available for the CDF. However, note that:

$$\int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \sqrt{2\pi}$$

Standard Gaussian PDF/CDF



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.4

9.0.2 Standard Gaussian Distribution (contd.)

- Properties: symmetry of PDF, symmetry of tail areas, of Z and $-Z$

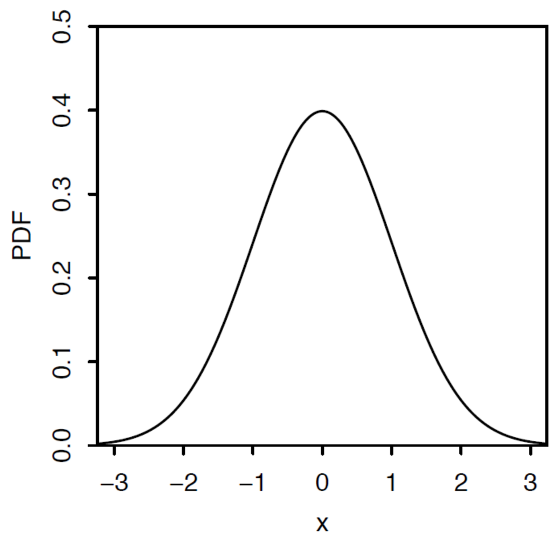
Mean: $$E\{Z\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz = 0$$

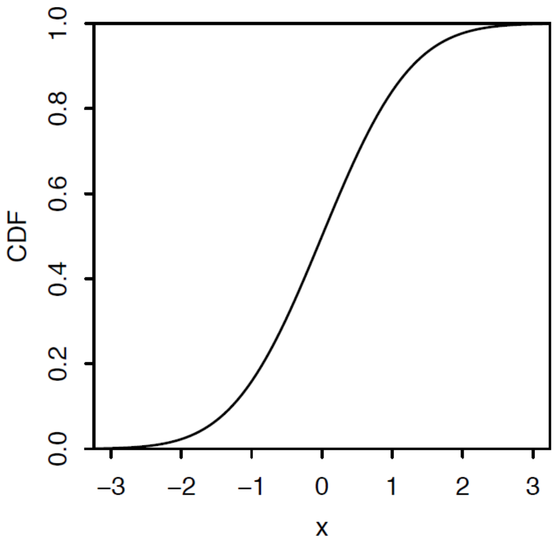
Variance *: $$Var\{Z\} = E\{Z^2\} - (E\{Z\})^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{\frac{-z^2}{2}}\, dz =$$

$$= \frac{2}{\sqrt{2\pi}} \left(-z e^{-z^2/2} \Big|_0^{\infty} + \int_0^{\infty} e^{\frac{-z^2}{2}}\, dz \right) = \frac{2}{\sqrt{2\pi}} \left(0 + \frac{\sqrt{2\pi}}{2} \right) = 1$$

(integrating by parts)

Standard Gaussian PDF/CDF





J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.4

**Using 8.3.3 LOTUS (W8)*

9.0.2 Gaussian Distribution

- Gaussian (or Normal) distribution with any mean μ and variance σ: location-scale transformation of the standard Normal

$$X = \mu + \sigma Z$$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

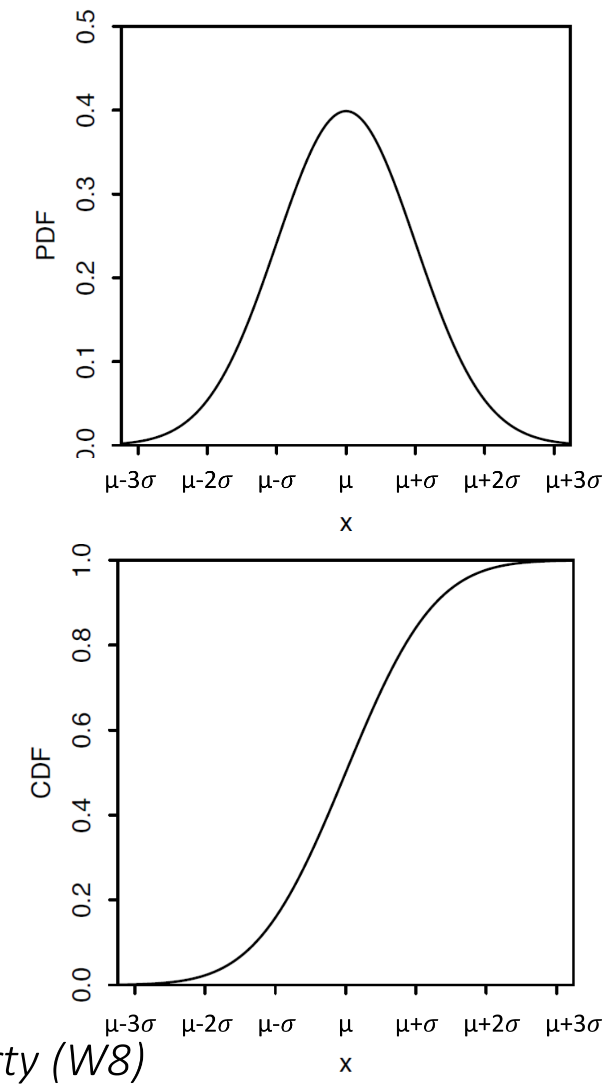
Mean*: $E\{X\} = E\{\mu + \sigma Z\} = E\{\mu\} + \sigma E\{Z\} = \mu$

Variance**: $Var\{X\} = Var\{\mu + \sigma Z\} = Var\{\sigma Z\} = \sigma^2 Var\{Z\} = \sigma^2$

- Standardisation process (from X back to Z):

$$\text{for } X \sim \mathcal{N}(\mu, \sigma^2), \qquad \frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Gaussian PDF/CDF



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.4

*Using linearity property (W8)

** Using 8.3.4 (W8)

9.0.2 Gaussian Distribution (contd.)

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- General Gaussian CDF $F(x)$ and PDF $f(x)$:

$$\text{CDF:} \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

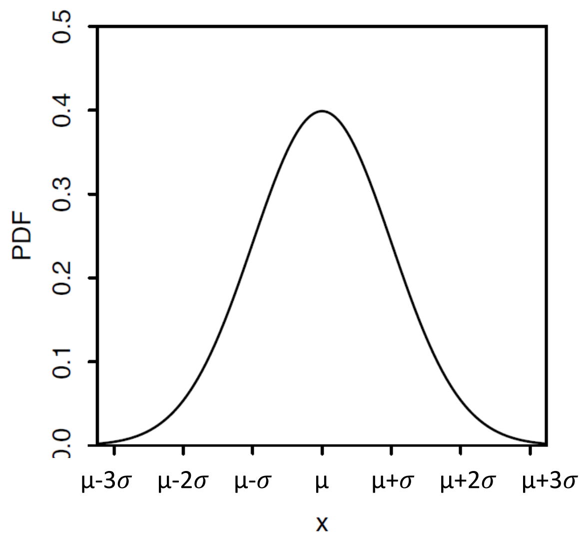
$$\text{PDF:} \quad f(x) = \varphi\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

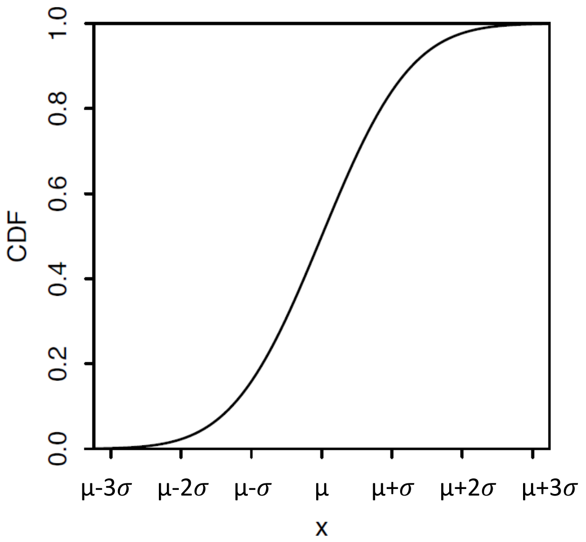
- *Proof:*

$$F(x) = P\{X \leq x\} = P\left\{\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) = \varphi\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma}exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Gaussian PDF/CDF





J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.4

9.0.2 Gaussian Distribution (contd.)

- Important properties – if $X \sim \mathcal{N}(\mu, \sigma^2)$,

$$P\{|X-\mu| < \sigma\} \approx 0.68$$

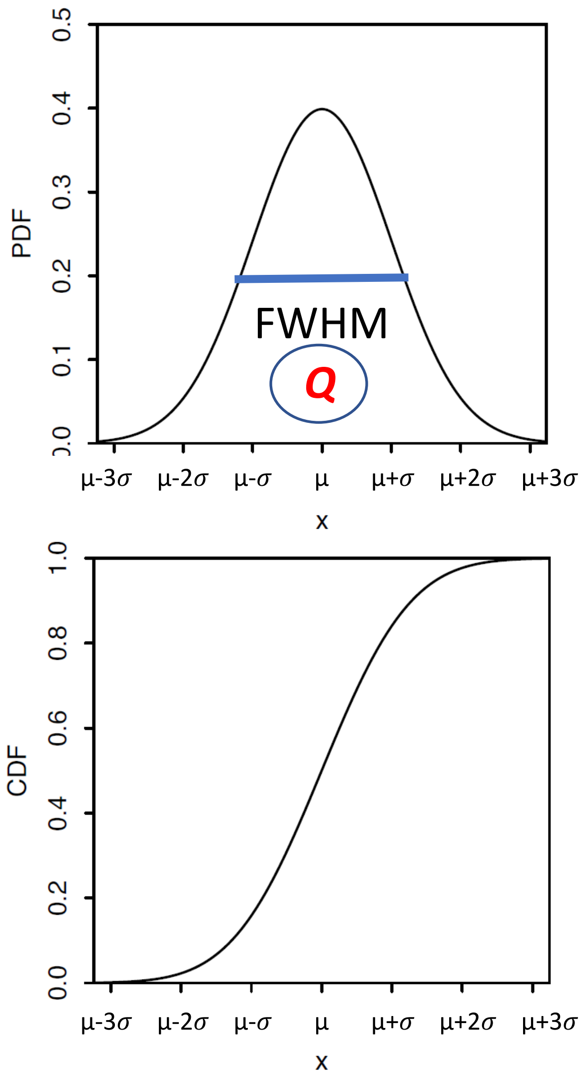
$$P\{|X-\mu| < 2\sigma\} \approx 0.95$$

$$P\{|X-\mu| < 3\sigma\} \approx 0.997$$

$$\text{Full Width Half Maximum (FWHM)} = P\{|X-\mu| < 1.175\sigma\}$$

$$FWHM = 2\sqrt{2\ln 2}\,\sigma \approx 2.355\,\sigma$$

Gaussian PDF/CDF

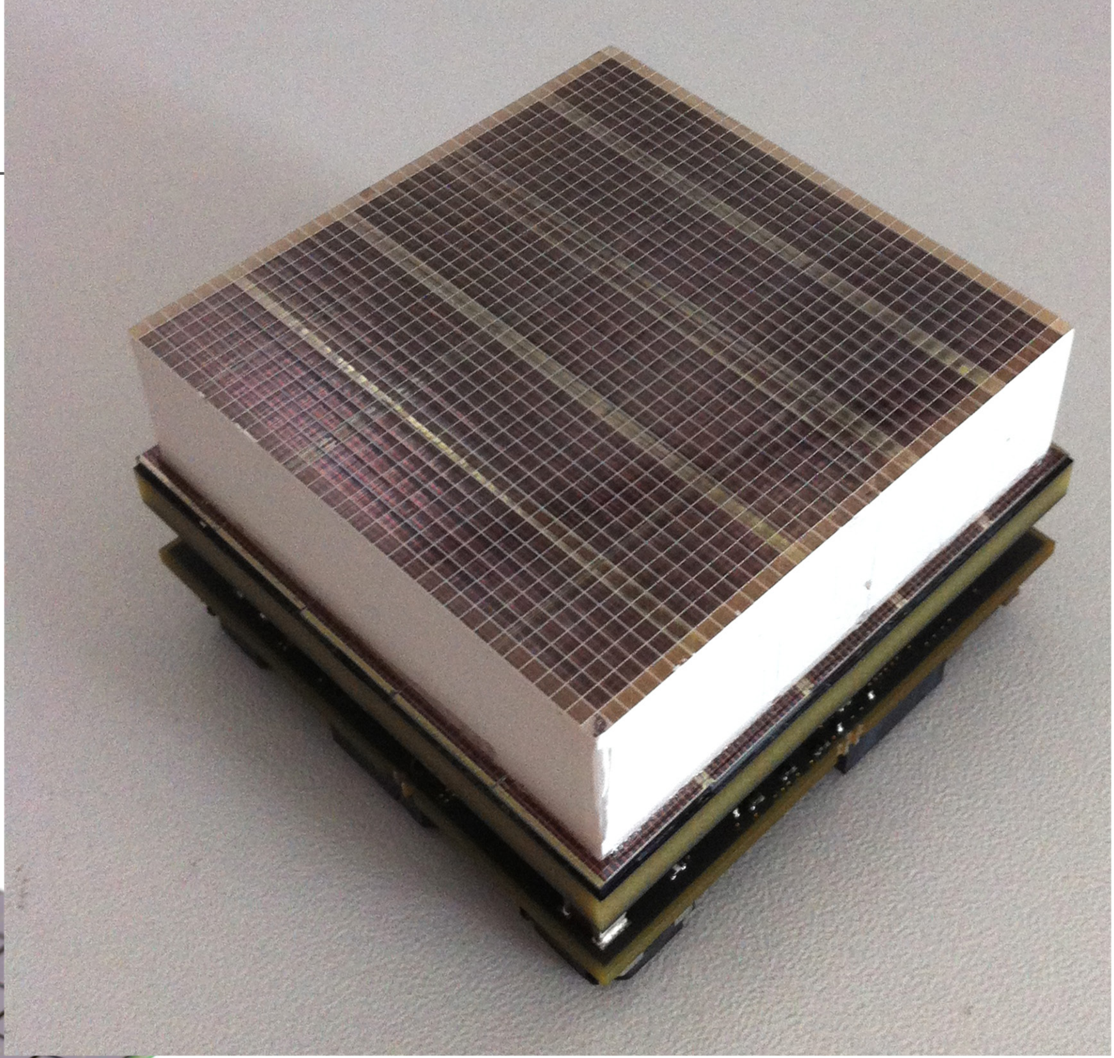


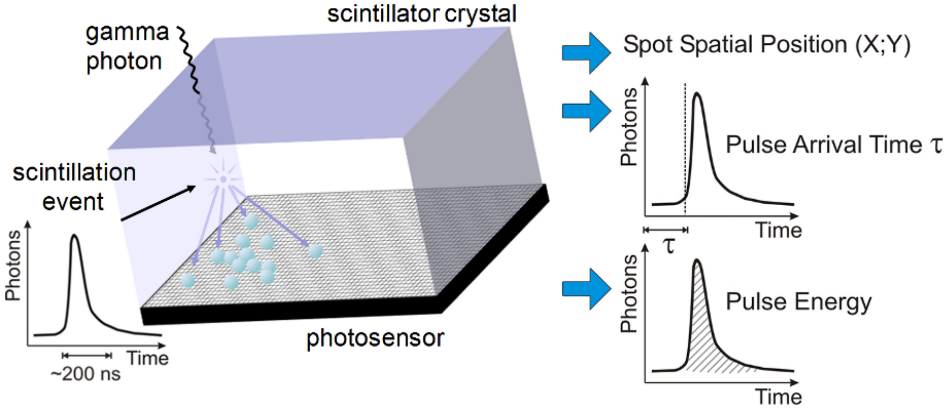
J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.4

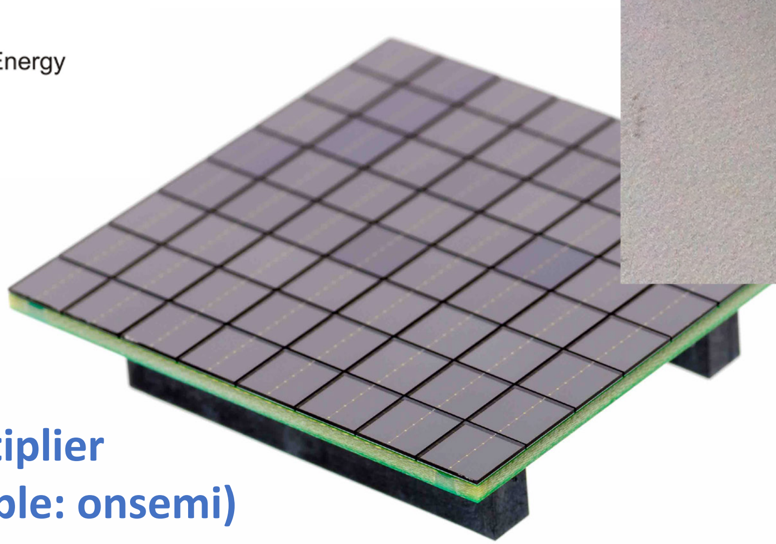
9.0.2 Gaussian Distribution – Example 1

Example of complete PET detection module

Scintillating crystal (LYSO)



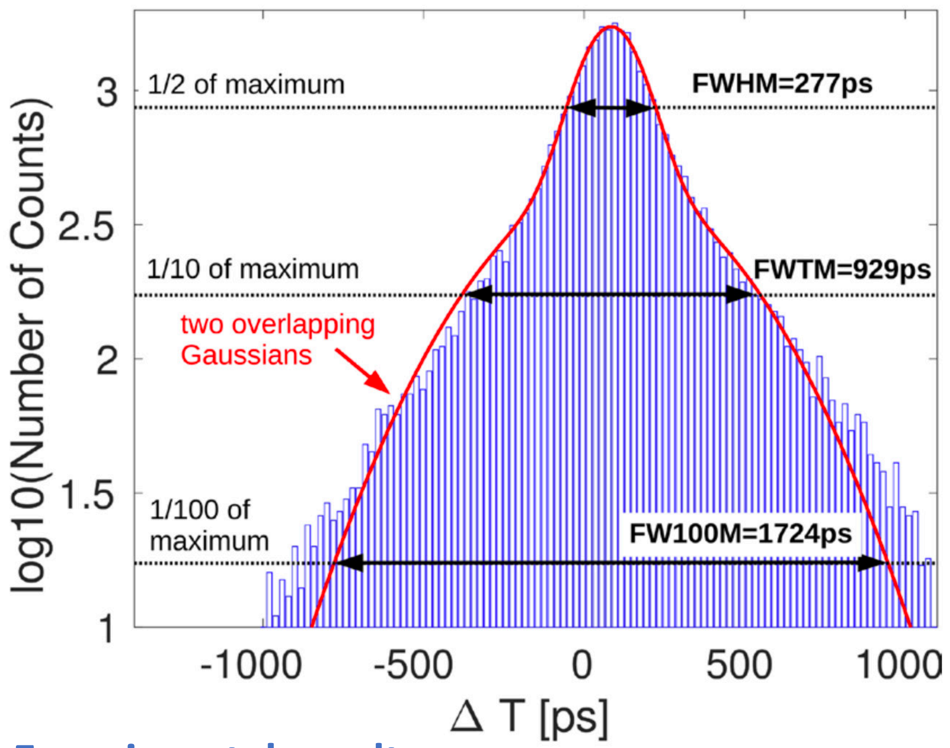




Silicon photomultiplier (SiPM) tile (example: onsemi)

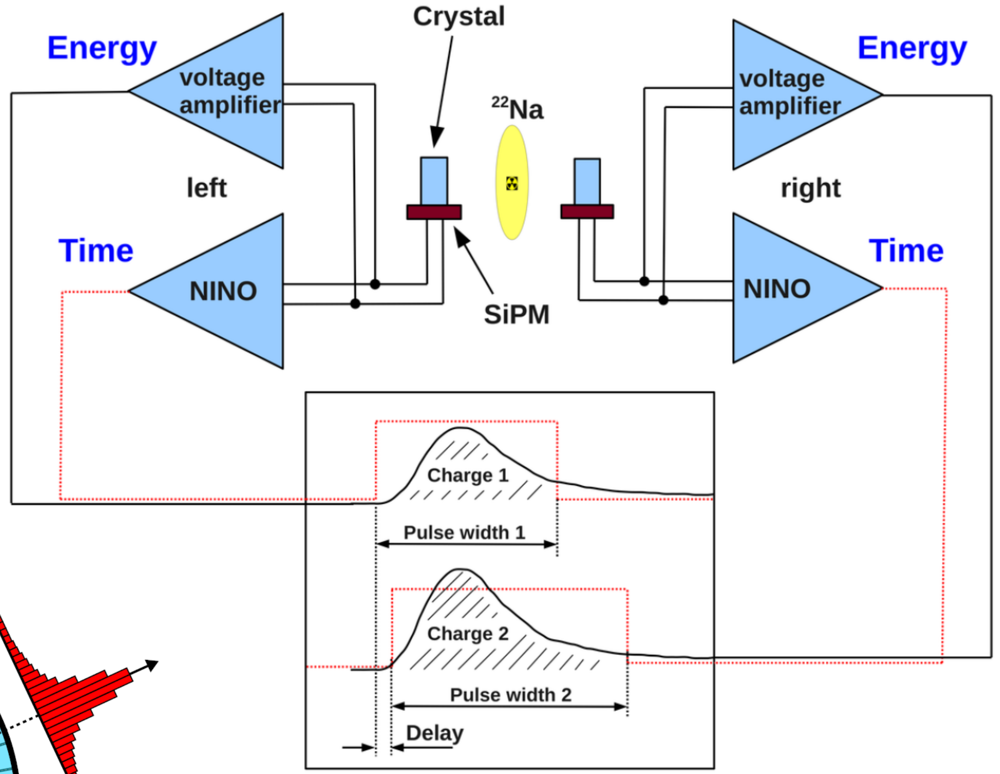
R. Walker *et al.*, IISW, 2013

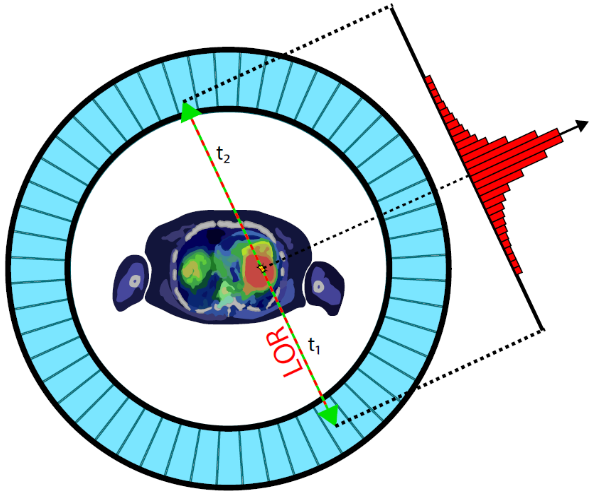
9.0.2 Gaussian Distribution – Example 1



Experimental results
(ΔT = Coincidence Time Resolution = T_2-T_2)

Simplified experimental set-up





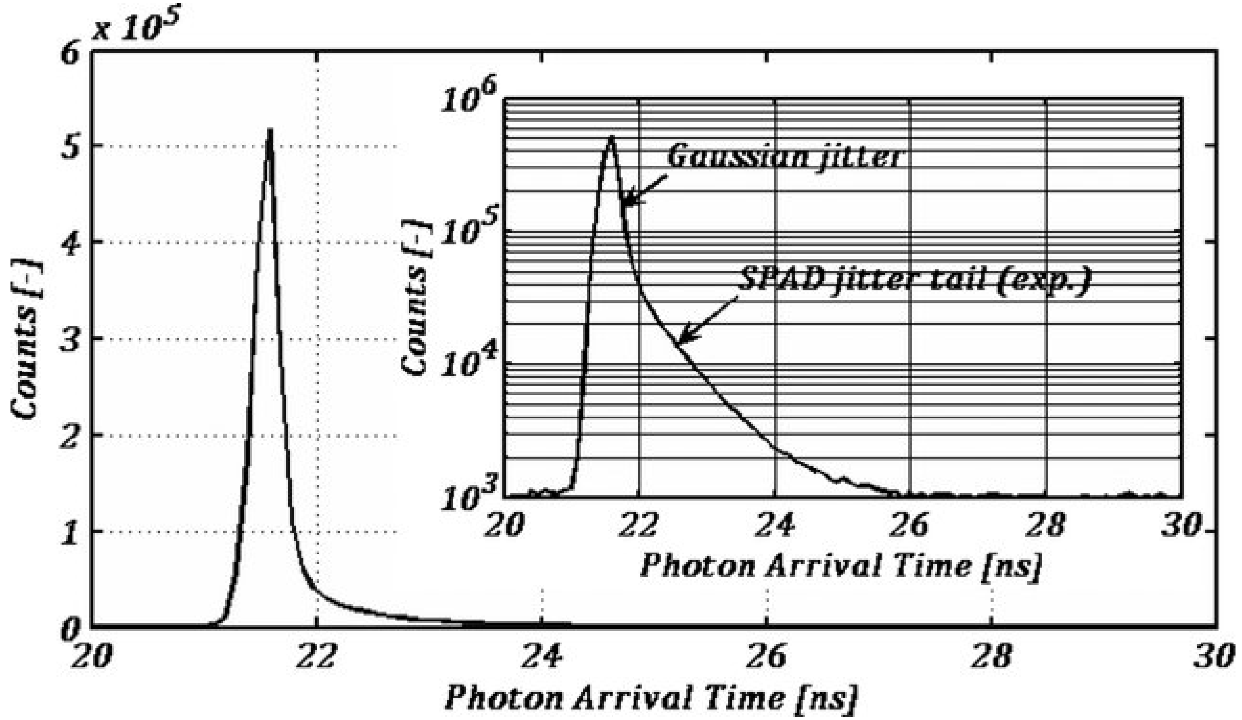
See also slide 27

📖 F. Gramuglia, *EPFL Thèse 8720* (2022).

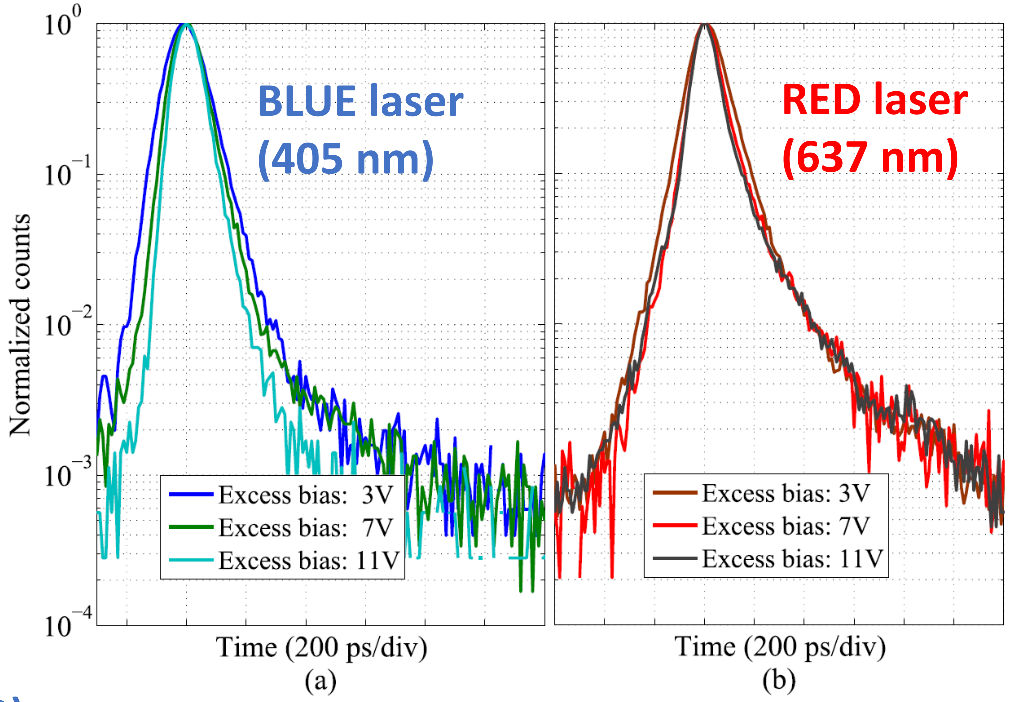
📖 S. Gundacker *et al.*, *Experimental time resolution limits of modern SiPMs and TOF-PET detectors exploring different scintillators and Cherenkov emission*, PMB 65 (2020).

📖 S. Gundacker *et al.*, *Time of flight positron emission tomography towards 100ps resolution with L(Y)SO: an experimental and theoretical analysis*, JINST 8 (2013).

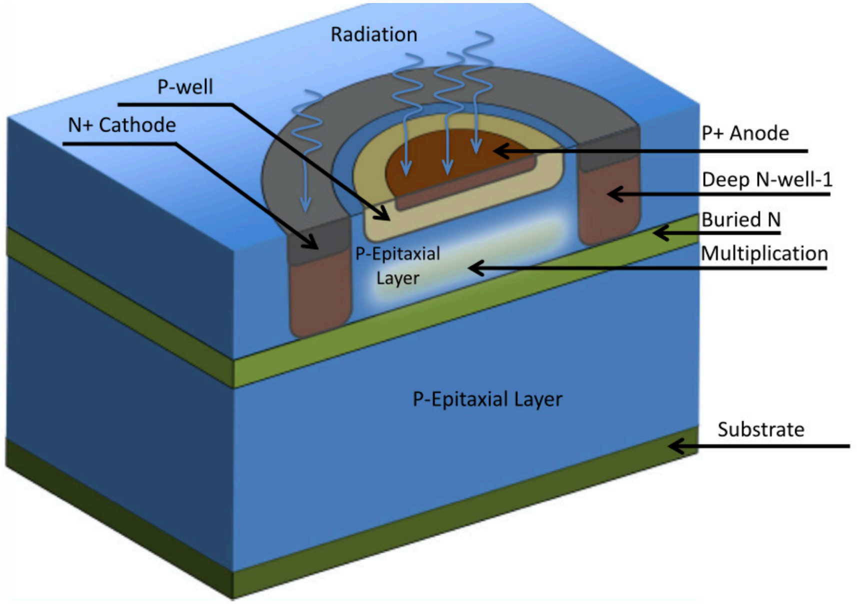
9.0.2 Gaussian Distribution – Example 2



A)



B)



(A) Non-Gaussian behavior – exponential tail – of the SPADs timing uncertainty (jitter noise) due to carrier diffusion -> (B) revised junction design

📖 C. Veerappan & E. Charbon, *A Low Dark Count p-i-n Diode Based SPAD in CMOS Technology,* IEEE TED 63 (2016).

📖 A. Ulku *et al., A 512×512 SPAD Image Sensor With Integrated Gating for Widefield FLIM,* IEEE JSTQE 25 (2019).

📖 C. Niclass *et al., A 128×128 Single-Photon Image Sensor With Column-Level 10-Bit Time-to-Digital Converter Array.* IEEE JSSC 43 (2008).

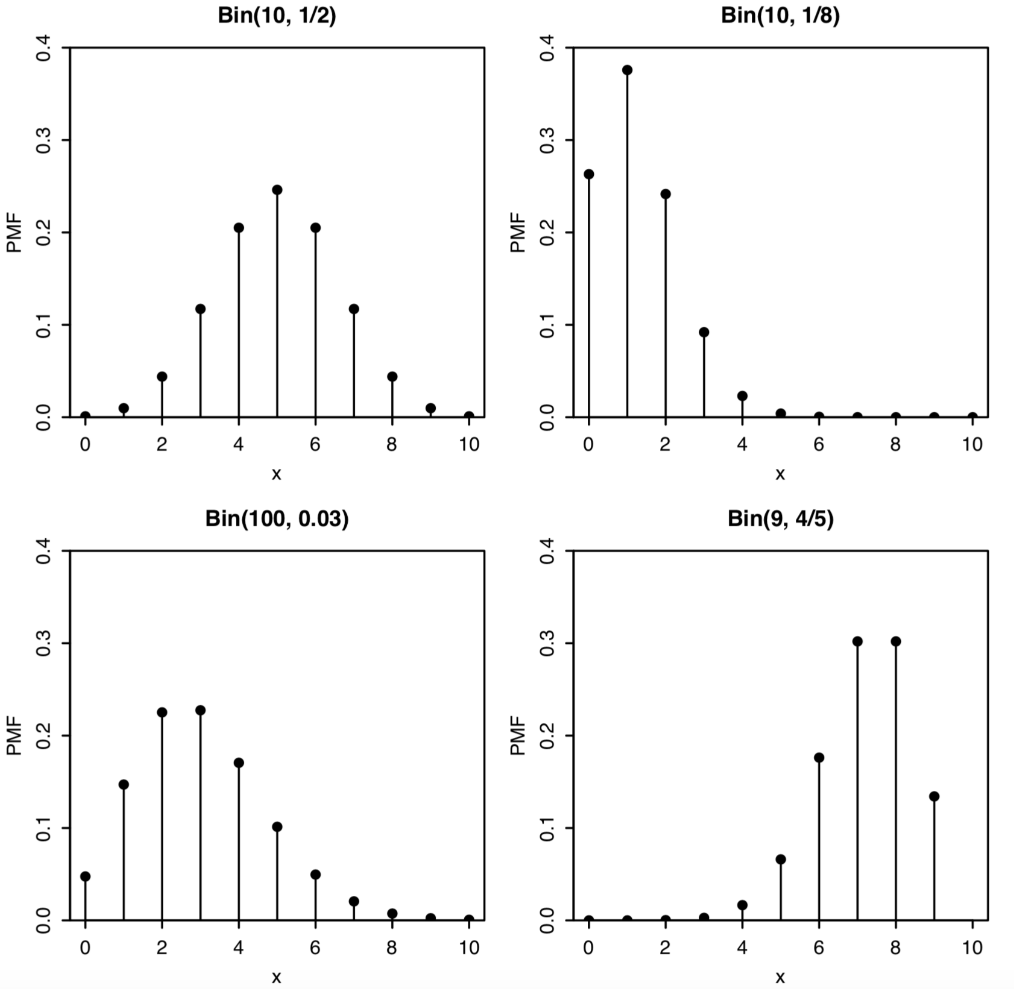
9.0.3 Binomial Distribution

- Suppose that n *independent* Bernoulli trials are performed. Let p be the probability of success, $1-p$ the probability of failure, X (RV) the number of successes.

- The distribution of X is called binomial distribution $\mathrm{Bin}(n,p)$ with parameters n and p if:

PMF: $$P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

Mean: $$E\{X\} = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 3.3

9.0.3 Binomial Distribution

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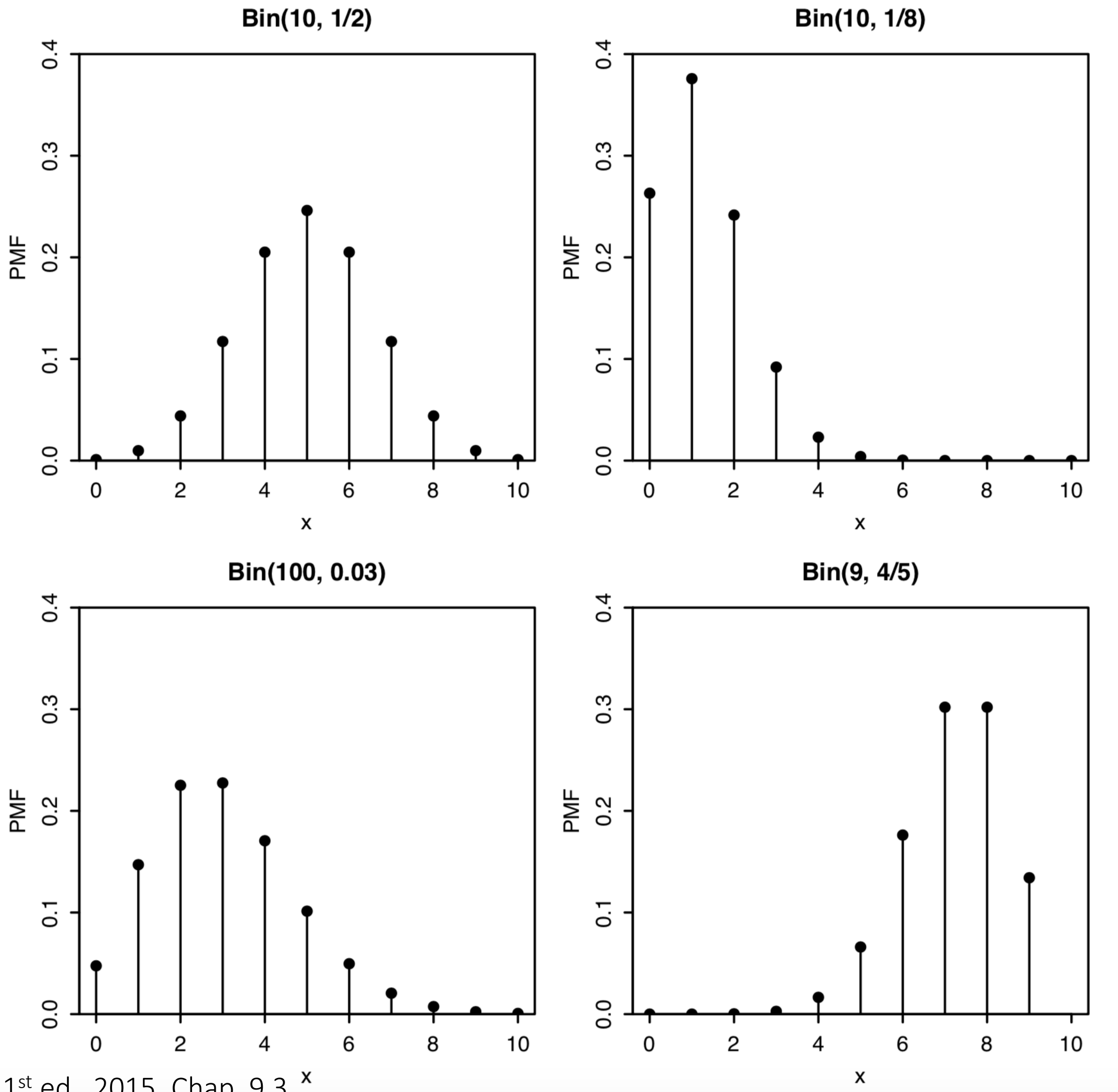


J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 9.3

9.0.4 Poisson Distribution

- Definition: a random variable $X \sim \mathrm{Pois}(\lambda)$ has a Poisson distribution with parameter λ if its PMF:

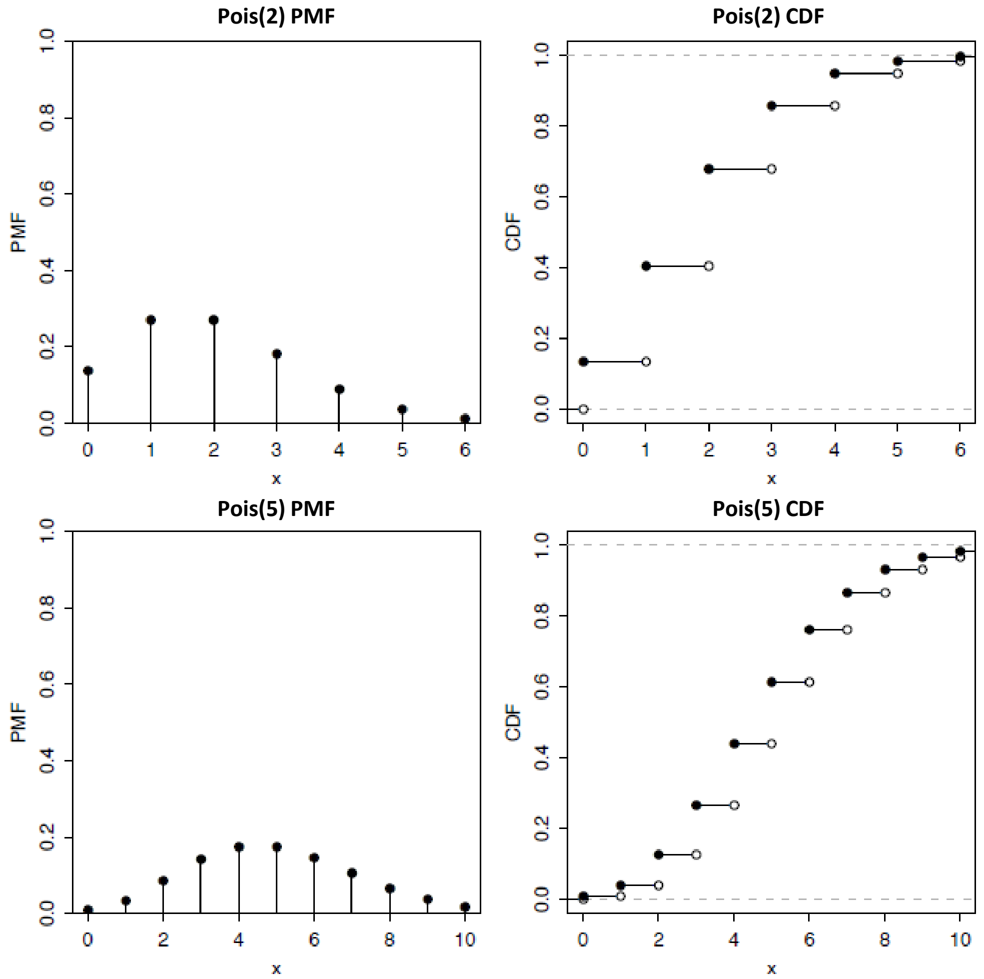
$$\text{PMF:} \quad P\{X = k\} = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0,1,2,\ldots$$

$$\text{Mean:} \quad E\{X\} = e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = \lambda$$

$$\text{Variance:} \quad Var\{X\} = E\{X^2\} - (E\{X\})^2 =$$

$$= \lambda(1+\lambda) - \lambda^2 = \lambda$$

$$\text{NB:} \quad \text{Taylor series:} \sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{\lambda}$$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 4.7

9.0.4 Poisson Distribution (contd.)

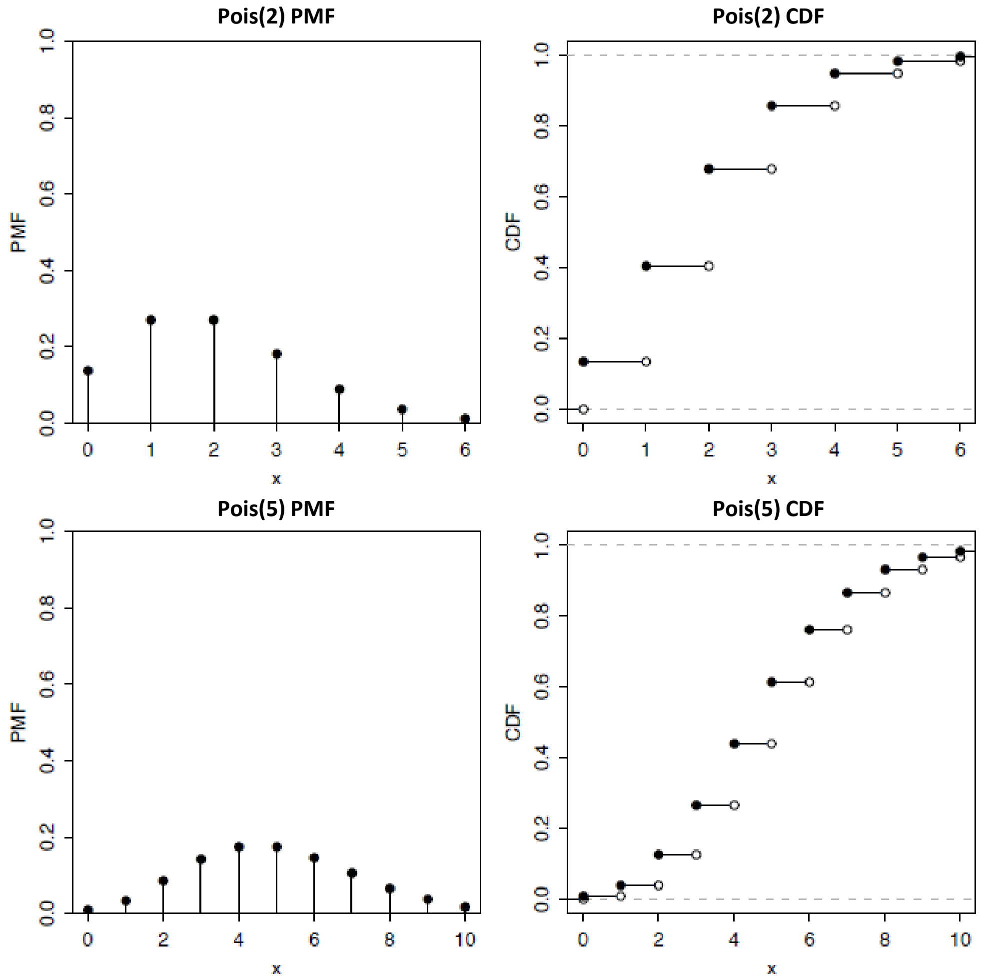
- The Poisson distribution has the following properties:

  1. If $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$ and X and Y are independent, then the distribution of

  $$X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$$

  2. If $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$ and X and Y are independent, then the conditional distribution of X given $X + Y = n$ is:

  $$P(X = k | X + Y = n) \sim \text{Bin}(n, \lambda_1/(\lambda_1 + \lambda_2))$$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 4.7

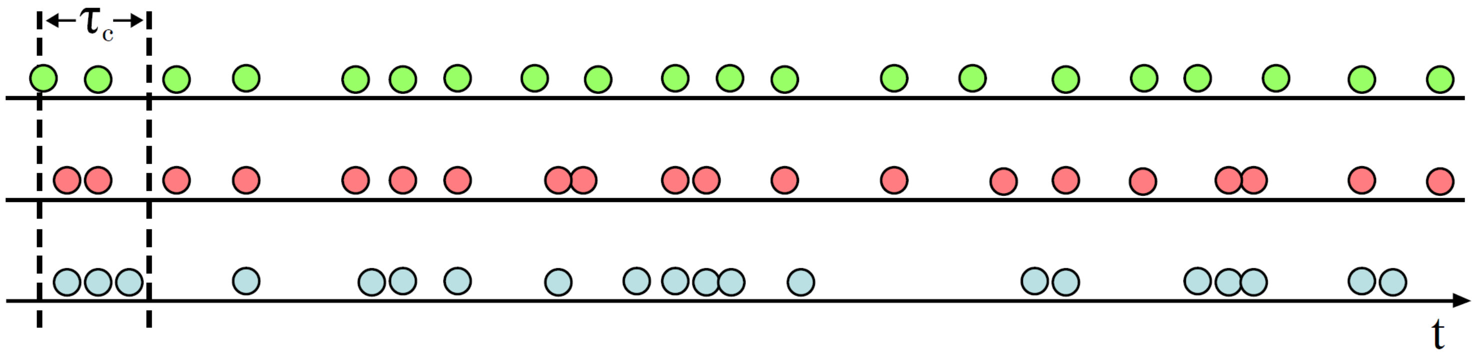
Poisson Distribution vs. Light Sources

- Non-classical light: Sub-Poissonian -> antibunched (anticorrelated)
- Coherent light source (Laser): Poissonian, random spacing (uncorrelated)
- Thermal Light: Super-Poissonian, Bose-Einstein distribution with zero counts as most probable count (bunched, positively correlated)

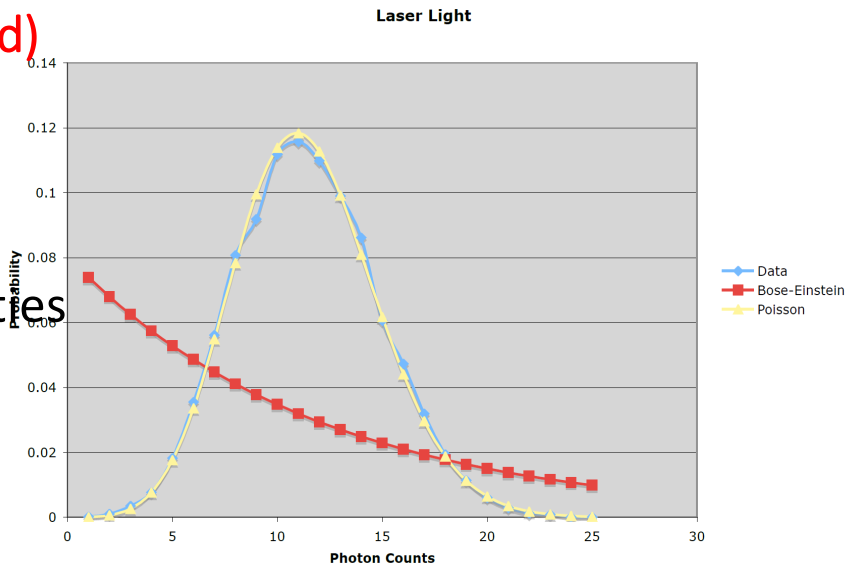
However, in practice it defaults to Gaussian due to the very low coherence time, O(ps), and the corresponding experimental difficultes

Experimentally one can use pseudothermal light*.

https://demonstrations.wolfram.com/PhotonNumberDistributions/



Photon detections as function of time for a) antibunched, b) random, and c) bunched light



**E.g.* scattering of a laser beam on a rotating ground glass disc

http://physics.gu.se/~tfkhj/lecture_X_differential_transmission-2.pdf
https://www.stmarys-ca.edu/sites/default/files/attachments/files/GriderJordanFinalReport_0.pdf

Poisson Distribution vs. Light Sources

$\bar{n}$ = average photon number

- Non-classical light: Sub-Poissonian

$$\boldsymbol{\sigma < \sqrt{\bar{n}}}$$

- Coherent light source (Laser): Poissonian

$$\boldsymbol{P(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \sigma = \sqrt{\bar{n}}}$$

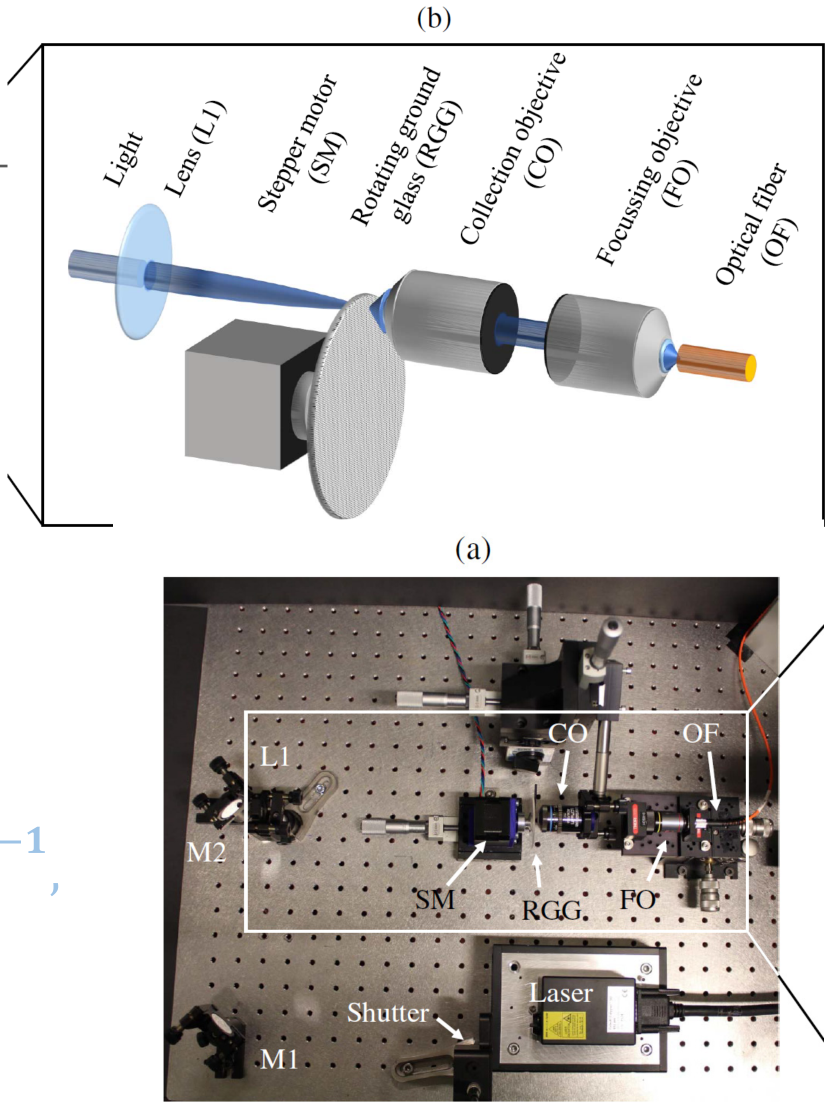
For large photon numbers, the relative fluctuations $\boldsymbol{\sigma/\bar{n}}$ tend to 0

- Thermal Light: Super-Poissonian, Bose-Einstein distribution

$$\boldsymbol{P(n) = \left(1 - e^{-\hbar\omega/k_B T}\right) e^{-n\hbar\omega/k_B T} = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}, \bar{n} = \left(e^{\hbar\omega/k_B T} - 1\right)^{-1},}$$

$$\boldsymbol{\sigma = \sqrt{\bar{n}^2 + \bar{n}}\ \ (for\ T \ll \tau_c) > \sqrt{\bar{n}}}$$

For large photon numbers, the relative fluctuations $\boldsymbol{\sigma/\bar{n}}$ tend to 1



Pseudothermal light source

Advanced Lab Course (F-Praktikum), Exp. 45, *Photon Statistics*, v. Aug. 21 2017

http://physics.gu.se/~tfkhj/lecture_X_differential_transmission-2.pdf

https://www.stmarys-ca.edu/sites/default/files/attachments/files/GriderJordanFinalReport_0.pdf

T. Stagner *et al.*, *Step-by-step guide to reduce spatial Coherence of laser light using a rotating ground glass diffuser*, OSA Applied Optics 56 (2017).

9.0.5 Exponential Distribution

- A continuous variable $Y \sim \text{Expo}(\lambda)$ has an Exponential distribution with parameter λ if:

$$\text{PDF:} \quad f_Y(y) = \lambda e^{-\lambda y}, \qquad y > 0$$

$$\text{CDF:} \quad F_Y(y) = 1 - e^{-\lambda y}, \qquad y > 0$$

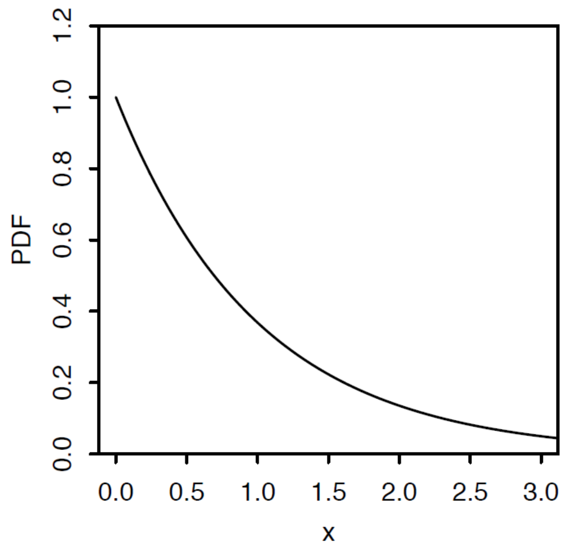
- If we start from $X \sim Expo(1)$:

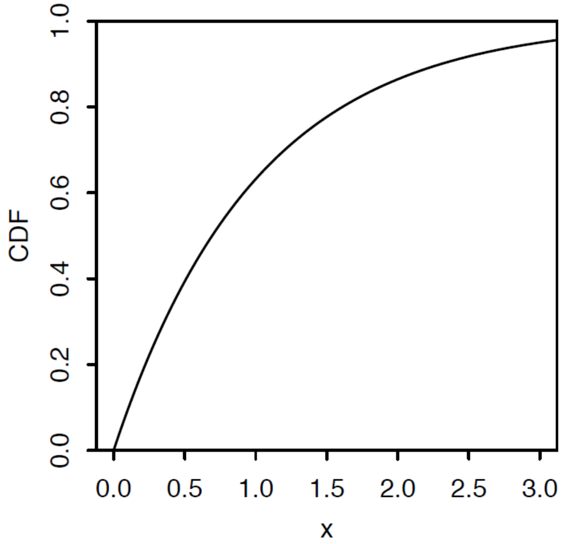
$$E\{X\} = \int_0^\infty x e^{-x}\, dx = 1$$

$$E\{X^2\} = \int_0^\infty x^2 e^{-x}\, dx = 2$$

$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = 1$$

Expo(1)





J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.5

9.0.5 Exponential Distribution (contd.)

- In general, for $Y = X/\lambda \sim \text{Expo}(\lambda)$ (scaling), we get:

$$\text{Mean:} \quad E\{Y\} = \frac{1}{\lambda} E\{X\} = \frac{1}{\lambda}$$

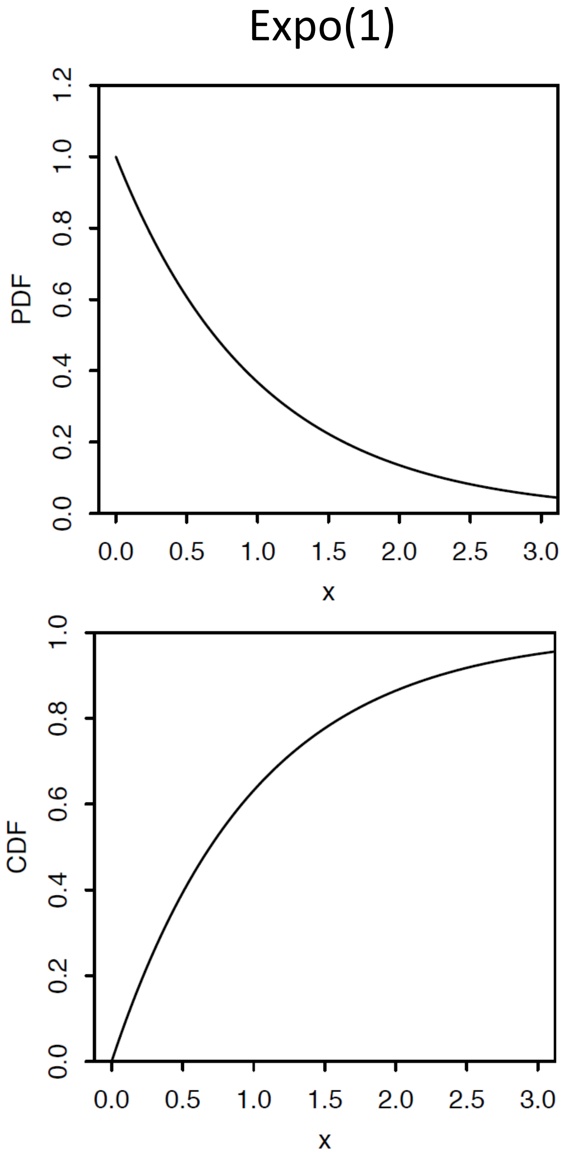
$$\text{Variance:} \quad Var\{Y\} = \frac{1}{\lambda^2} Var\{X\} = \frac{1}{\lambda^2}$$

- Recap: *«An* $\text{Expo}(\lambda)$ *RV represents the waiting time for the first success in continuous time; the parameter* λ *can be interpreted as the rate at which successes arrive."*

- Memoryless property: *«conditional on our having waited a certain amount of time (s) without success, the distribution of the remaining wait time (t) is exactly the same as if we hadn't waited at all.»*

$$\text{Memoryless:} \quad P\{Y \geq s + t | Y \geq s\} = P\{Y \geq t\}$$

Ex



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.5

9.0.5 Exponential Distribution (contd.)

- Memoryless property: *«conditional on our having waited a certain amount of time (s) without success, the distribution of the remaining wait time (t) is exactly the same as if we hadn't waited at all.»*

$$\text{Memoryless:} \quad P\{Y \geq s + t | Y \geq s\} = P\{Y \geq t\}$$

$$e.g. P\{Y \geq 40 | Y \geq 30\} = P\{Y \geq 10\}$$

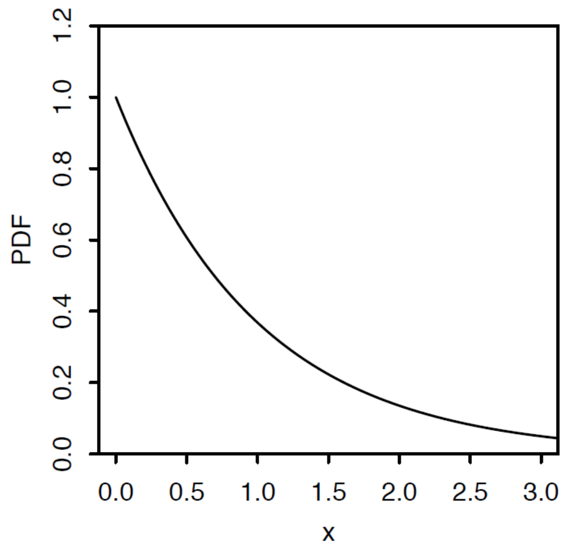
$$e.g. P\{Y \geq 70 | Y \geq 60\} = P\{Y \geq 10\}$$

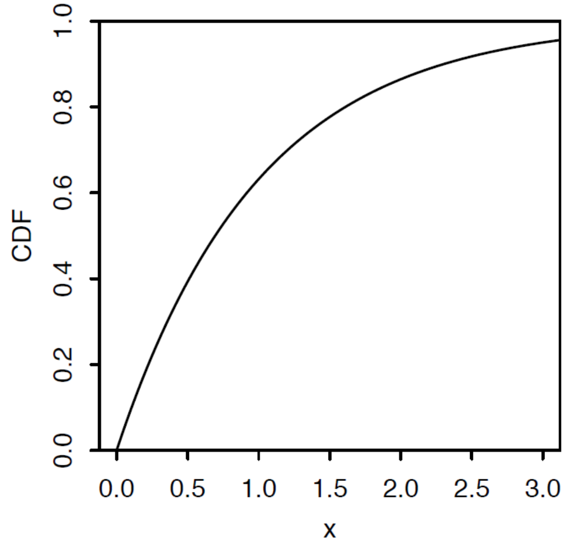
$$\text{PDF:} \quad f_Y(y) = \lambda e^{-\lambda y}, y_2 = y_1 + \Delta t$$

$$\frac{f_Y(y_2)}{f_Y(y_1)} = \frac{\lambda e^{-\lambda y_2}}{\lambda e^{-\lambda y_1}} = e^{-\lambda \, \Delta t} = constant$$

$$e.g. \frac{f_Y(y_2 = 4\,\lambda^{-1})}{f_Y(y_1 = 3\,\lambda^{-1})} = \frac{f_Y(y_2 = 2\,\lambda^{-1})}{f_Y(y_1 = 1\,\lambda^{-1})} = e^{-1} = constant$$

Expo(1)





J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.5

9.0.5 Exponential Distribution – Example 1 & 2

Radioactive decay

- Universal law of radioactive decay:
  - A nucleus has “no memory”
  - A nucleus does not age with the passage of time

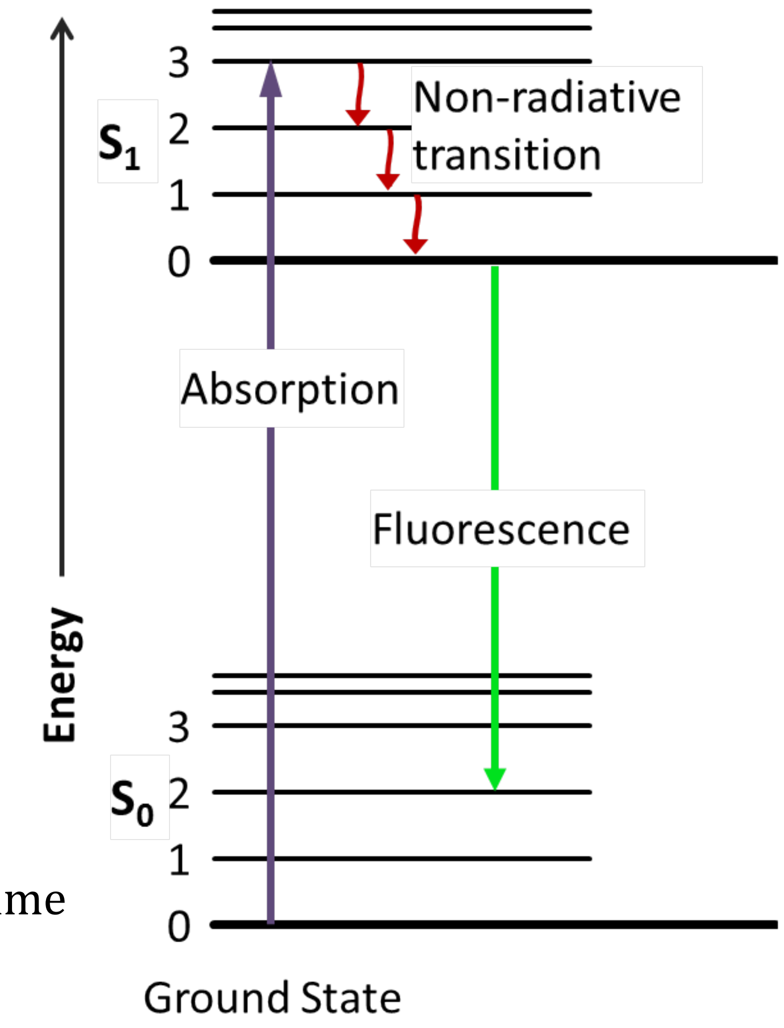
  -> a nucleus is equally likely to decay at any instant in time

  -> constant decay probability

Decay Law: $$\frac{dN}{dt} = -\lambda N \Rightarrow N(t) = N_0 e^{-\lambda t}$$

- NB: The number of decays in a given time interval in a radioactive sample is Poisson distributed…

Jablonski diagram



By Jacobkhed - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=19180813

Fluorescence lifetime

$$[S_1] = [S_1]_0 e^{-\Gamma t}$$

Q

S_1 = concentration of excited state molecules

Γ = decay rate = inverse of fluorescence lifetime = average length of time to decay from one state to another

EN Wikipedia *Radioactive_decay / Fluorescence*

9.0.5 Exponential Distribution – Example 3

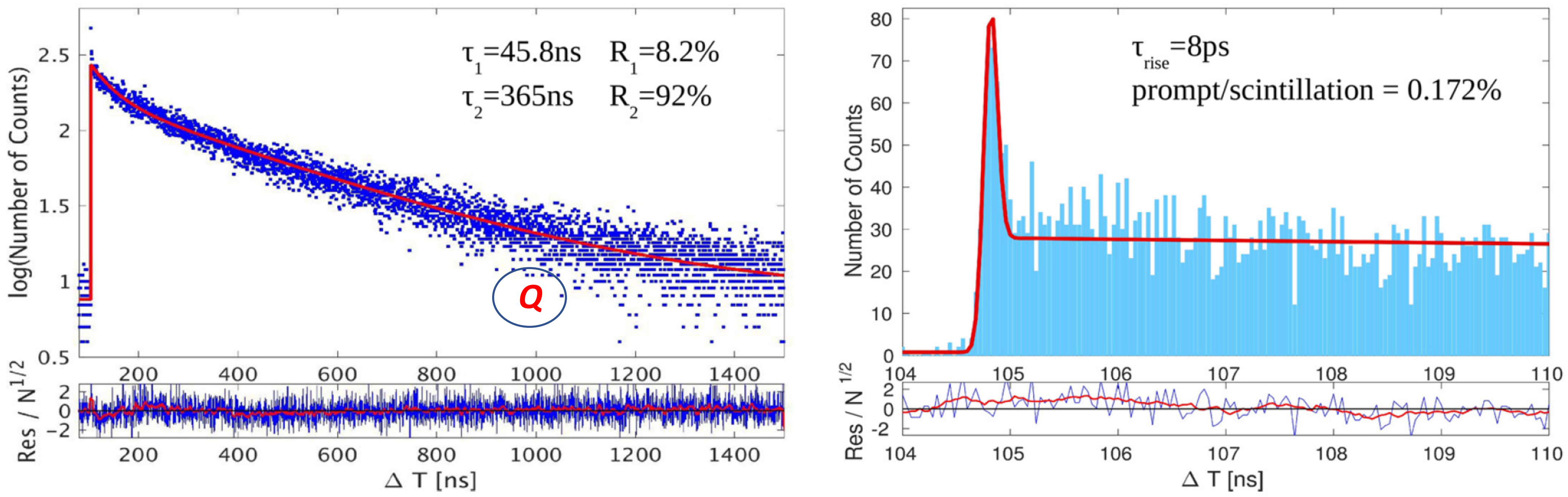


Figure 10. Scintillation decay and rise time of BGO measured with a time correlated single photon counting (TCSPC) setup using 511 keV annihilation gammas (Gundacker *et al* 2016b). The figure on the right hand side shows a pronounced Cherenkov peak at the onset of the scintillation emission with a relative abundance of 0.172% compared to the total amount of photons detected by the stop detector of the TCSPC setup.

Fast vs. "slow" scintillation photons in a heavy scintillating crystal

"Physical experiments are imprecise and generate errors handled by statistical methods."

(I. Vardi)

See also slide 14

Gundacker S, Auffray E, Pauwels K and Lecoq P *Measurement of intrinsic rise times for various L(Y)SO and LuAG scintillators with a general study of prompt photons to achieve 10 ps in TOF-PET.* IOP Phys. Med. Biol. 61 2802–37

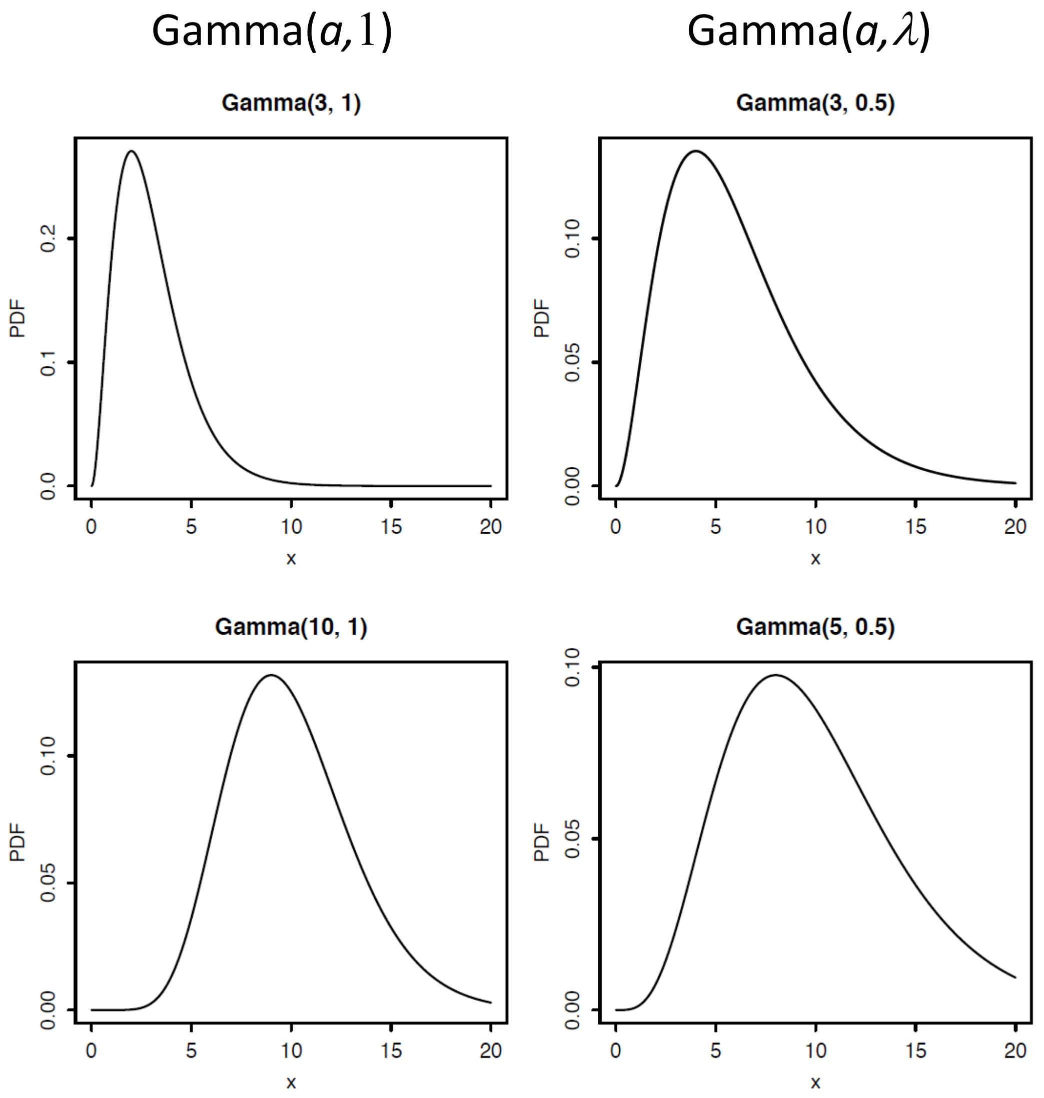
9.0.6 Gamma Distribution

S

- Let $\boldsymbol{X_1, X_2, \ldots, X_n}$ be $\boldsymbol{n}$ i.i.d. $\mathbf{Expo}(\boldsymbol{\lambda})$. Then:

$$\boldsymbol{Y = X_1 + \cdots + X_n \sim \mathbf{Gamma}(n, \lambda)}$$

- The Gamma is nothing else but the distribution obtained by summing up n independent exponential distributions.



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

9.0.6 Gamma Distribution (contd.)

S

- For the more general gamma distribution $Y = X/\lambda \sim \text{Gamma}(a, \lambda)$, by simple transformation, we obtain:

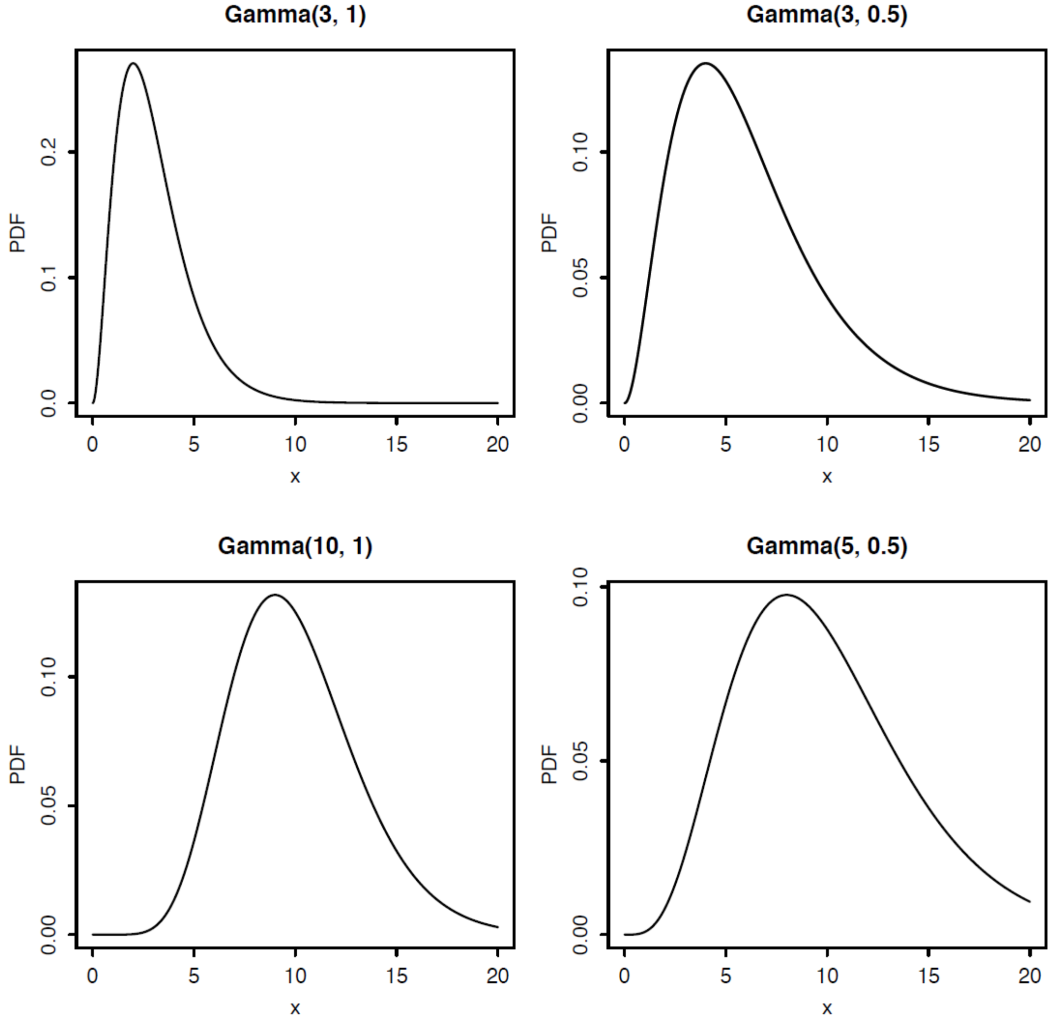
$$\text{Mean:} \quad E\{Y\} = \frac{1}{\lambda} E\{X\} = \frac{a}{\lambda}$$

$$\text{Second Moment:} \quad E\{Y^2\} = \frac{1}{\lambda^2} E\{X\} = \frac{a\,(a+1)}{\lambda^2}$$

$$\text{Variance:} \quad Var\{Y\} = \frac{1}{\lambda^2} Var\{X\} = \frac{a}{\lambda^2}$$

-> See Appendix A for details

$\text{Gamma}(a, 1)$ $\text{Gamma}(a, \lambda)$



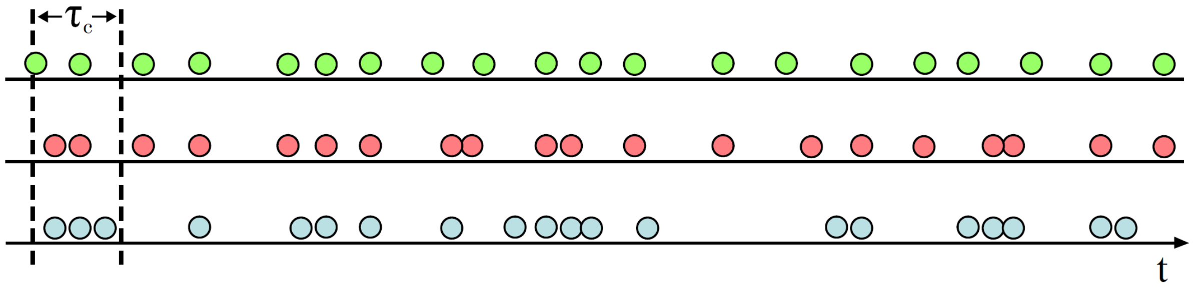
-> calculate mean/variance for some examples

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

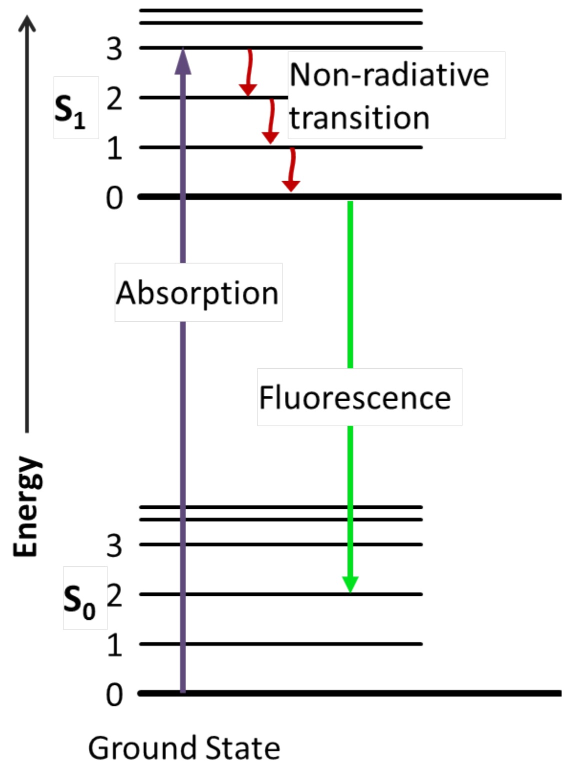
Take-home Messages/W9-1

- *Random Variables (RVs):*
  - Distributions: Uniform, Gaussian, Binomial
  - Distributions: Poisson ↔ Exponential

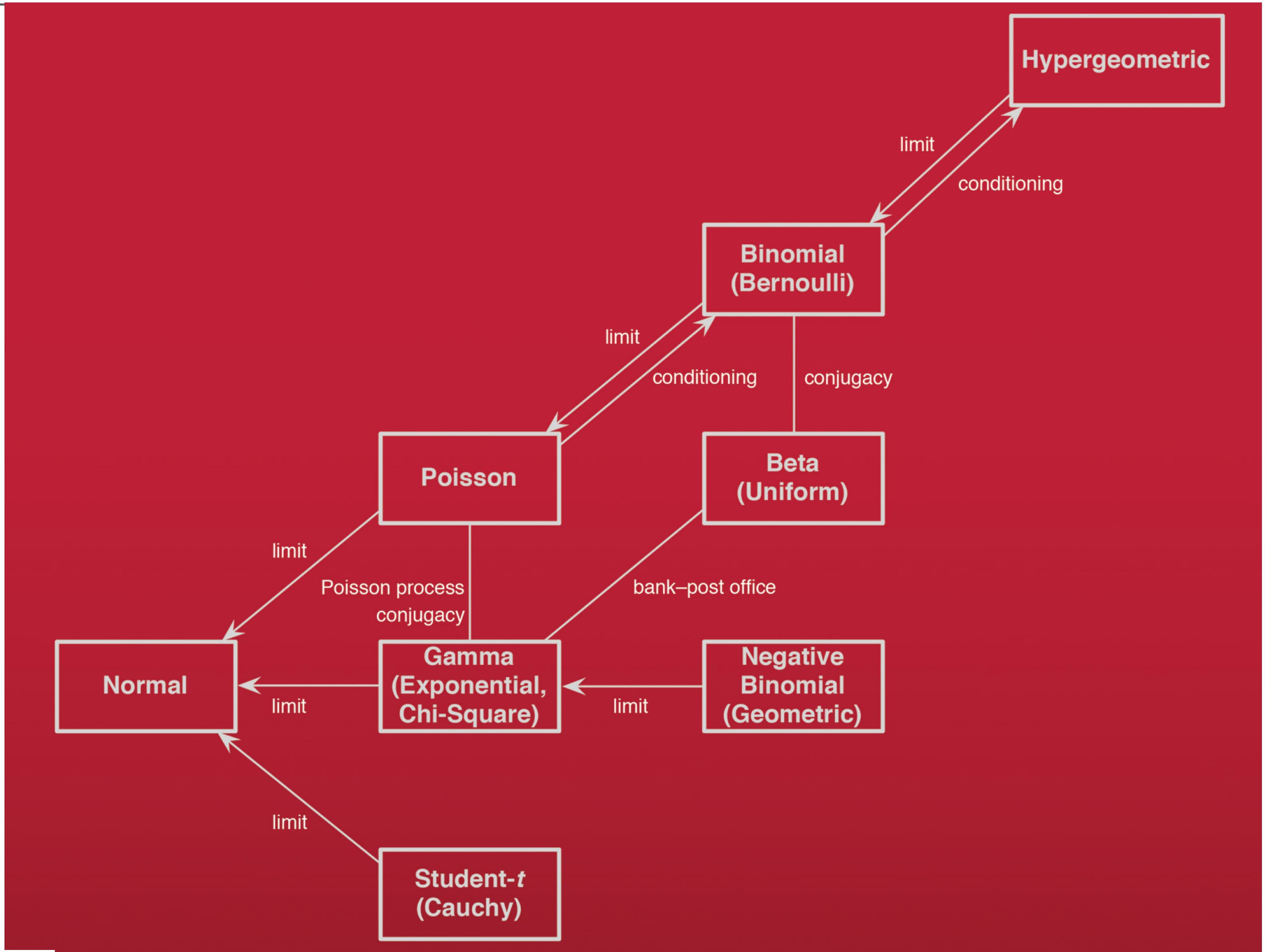
    … and their PDF, CDF, Mean, Variance
  - Practical examples!
    - Scintillation light (two crystals in coincidence) – combination of distributions ↔ experimental set-up
    - Timing jitter – combination of distributions ↔ experimental set-up
    - *Poisson Distribution vs. Light Sources*
    - Fluorescence lifetime & exponential decay
    - Scintillation light (one single crystal) ↔ experimental set-up



Photon detections as function of time for a) antibunched, b) random, and c) bunched light



Probability distributions – Connections & the Big Picture



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Cover page.

Outline

9.1.1 Random Process

- A Random (or stochastic) Process (RP) is a time-varying function that assigns the outcome of a random experiment to each time instant $X(t)$

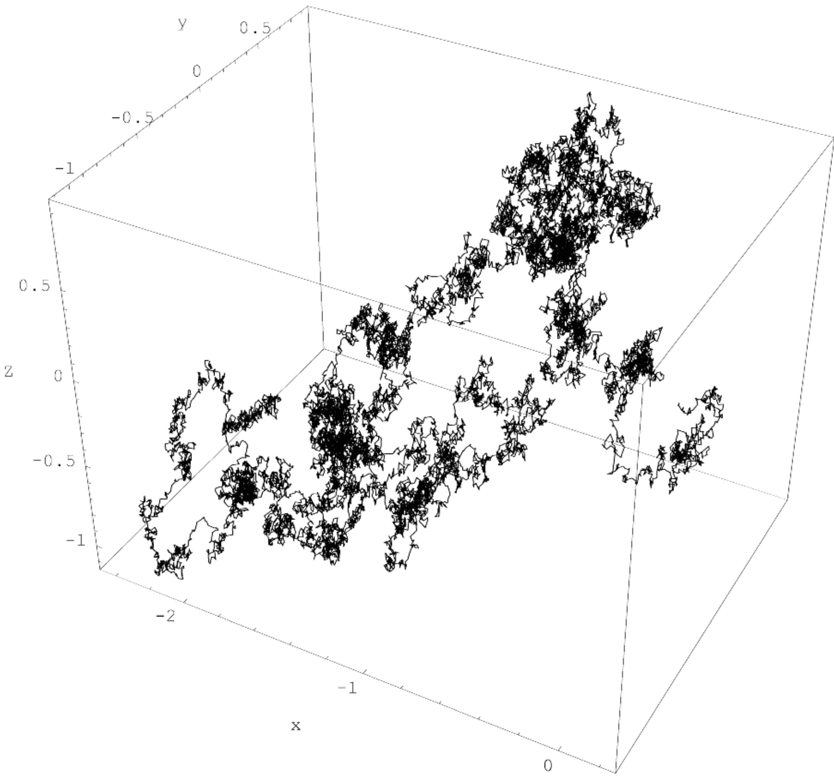
  Example: a current fluctuating due to thermal noise (-> Week 10), the growth of a bacterial population, the movement of a gas molecule [Wikipedia *Stochastic Process*]

- For fixed t, a Random Process is a Random Variable

- A Random Process can therefore be viewed as a **collection of an infinite number of Random Variables**. Given that $X_i = X(t_i)$:

$$\text{joint PDF:} \qquad f_X(X_1, X_2, \ldots, X_n, t_1, t_2, \ldots, t_n)$$

- A Random Process can be either continuous or discrete



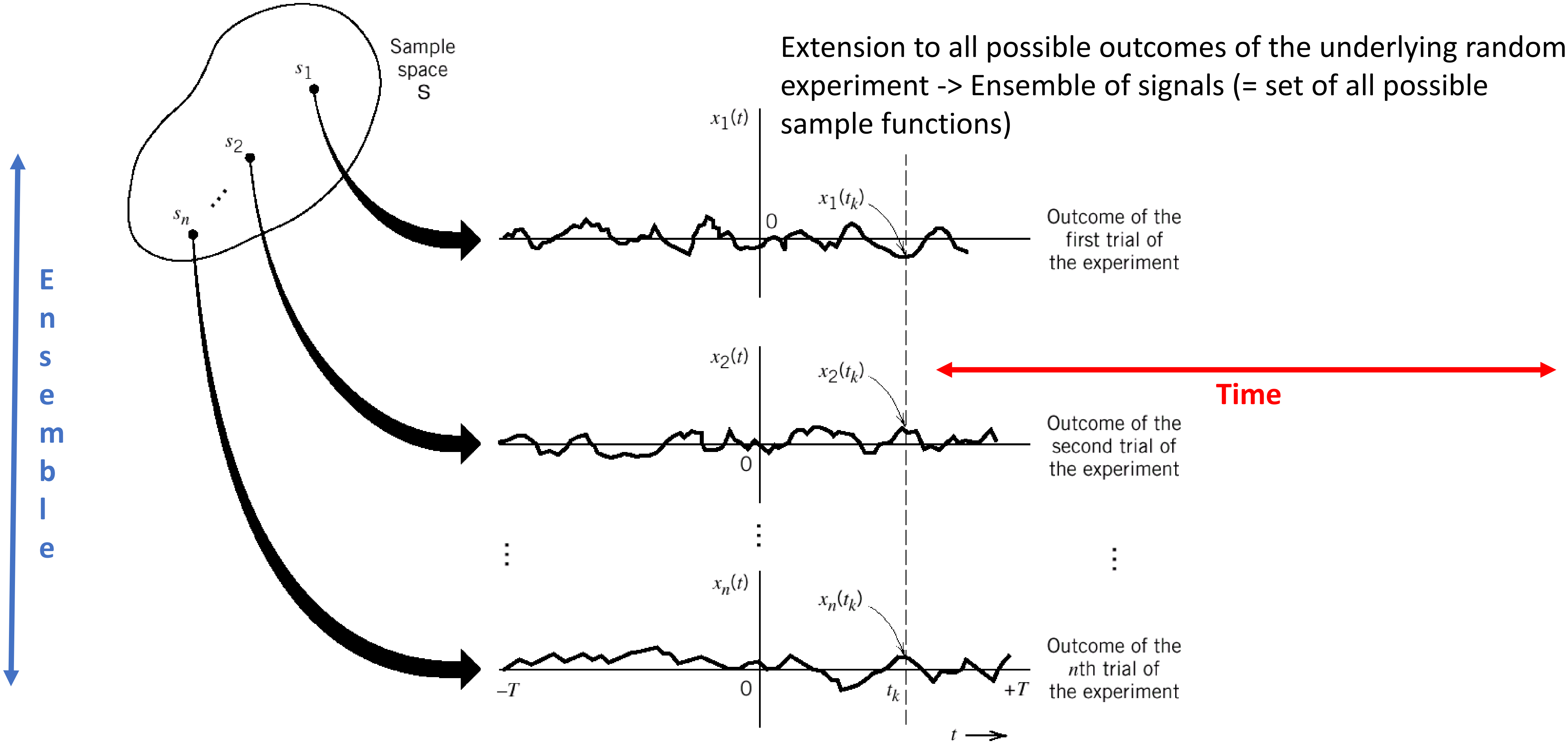
Original uploader was Sullivan.t.j at English Wikipedia. – 3D Brownian motion process. This mathematical image was created with Mathematica., CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=2249027

H. Bilgekul, Slides for the course “EE-461 Communication System II”, EMU

F. Farahmand, Slides for the course “CES 540 Digital Communication”, Ch. 6, SSU

9.1.1 Random Process – Example

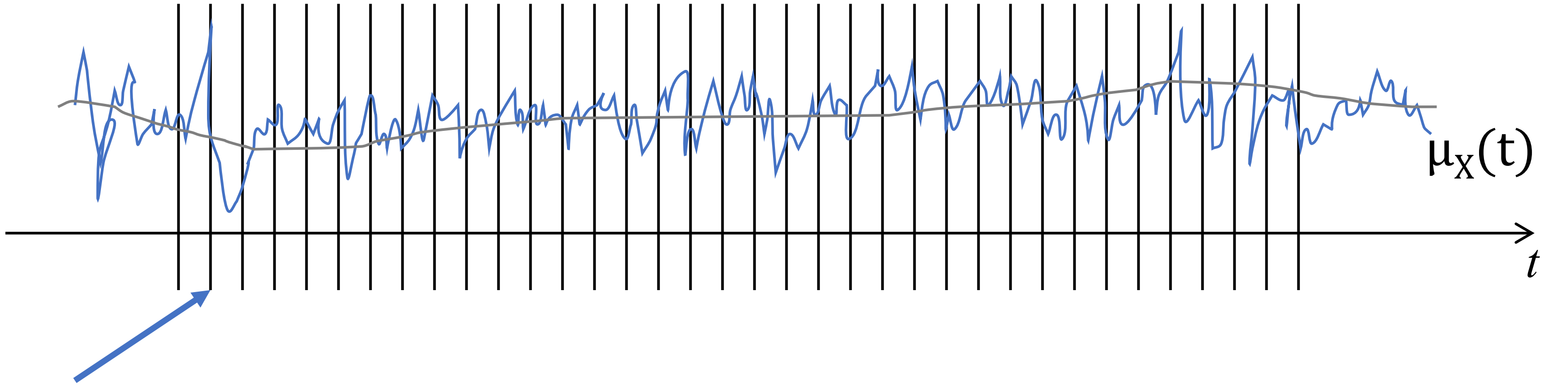
Extension to all possible outcomes of the underlying random experiment -> Ensemble of signals (= set of all possible sample functions)



H. Bilgekul, Slides for the course "EE-461 Communication System II", EMU   F. Farahmand, Slides for the course "CES 540 Digital Communication", Ch. 6, SSU

9.1.1 Random Process – Example

- Example: Noise is generally modeled as a random process, i.e. a collection of random variables, one for each time instant t in interval $]-\infty,+\infty[$



Fixed t: Random Process becomes a Random Variable

E. Charbon, "Image Sensors – ET 4390 Course Slides", Delft 2016

9.1.1 Random Process (contd.) – Characterization/1

- A Random Process is characterized by the same functions already explained for RVs, but which now depend on ***t***, i.e.:

CDF: $$F_X(x,t) = P\{X(t) \le x\}$$

$X(t)$= random variable at time t

PDF: $$f_X(x,t) = \frac{dF_X(x,t)}{dx}$$

Mean: $$m_X(t) = \overline{X(t)} = E\{X(t)\} = \int_{-\infty}^{\infty} x\, f_X(x,t)\, dx$$

Second Order Moment: $$\overline{X^2(t)} = E\{X^2(t)\} = \int_{-\infty}^{\infty} x^2\, f_X(x,t)\, dx$$

Variance: $$Var\{X(t)\} = E\left\{\left(X(t) - m_X(t)\right)^2\right\} = \int_{-\infty}^{\infty} \left(x - m_X(t)\right)^2 f_X(x,t)\, dx$$

Ensemble averages

H. Bilgekul, Slides for the course "EE-461 Communication System II", EMU

F. Farahmand, Slides for the course "CES 540 Digital Communication", Ch. 6, SSU

9.1.1 Random Process (contd.) – Characterization/2

- However, in order to characterize a RP, we need to introduce two more functions, e.g. to indicate how rapidly a RP changes in time:

$X(t_1)$= random variable at time t_1
$X(t_2)$= random variable at time t_2

$$\text{Auto}-\text{covariance:} \qquad C_{XX}(t_1, t_2) = Cov\{X(t_1), X(t_2)\}$$

$$\text{Auto}-\text{correlation:} \qquad K_{XX}(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\}$$

$$\text{NB:} \qquad C_{XX}(t_1, t_2) = E\{[X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)]\} = \\ = K_{XX}(t_1, t_2) - m_X(t_1) m_X(t_2)$$

NB: in general, the *autocorrelation* is the correlation of the signal with a delayed copy of itself (similarity between observations as a function of the time lag between them) [Wikipedia "autocorrelation"]

Cross-correlation: same but between two series

- In a similar way we can also define:

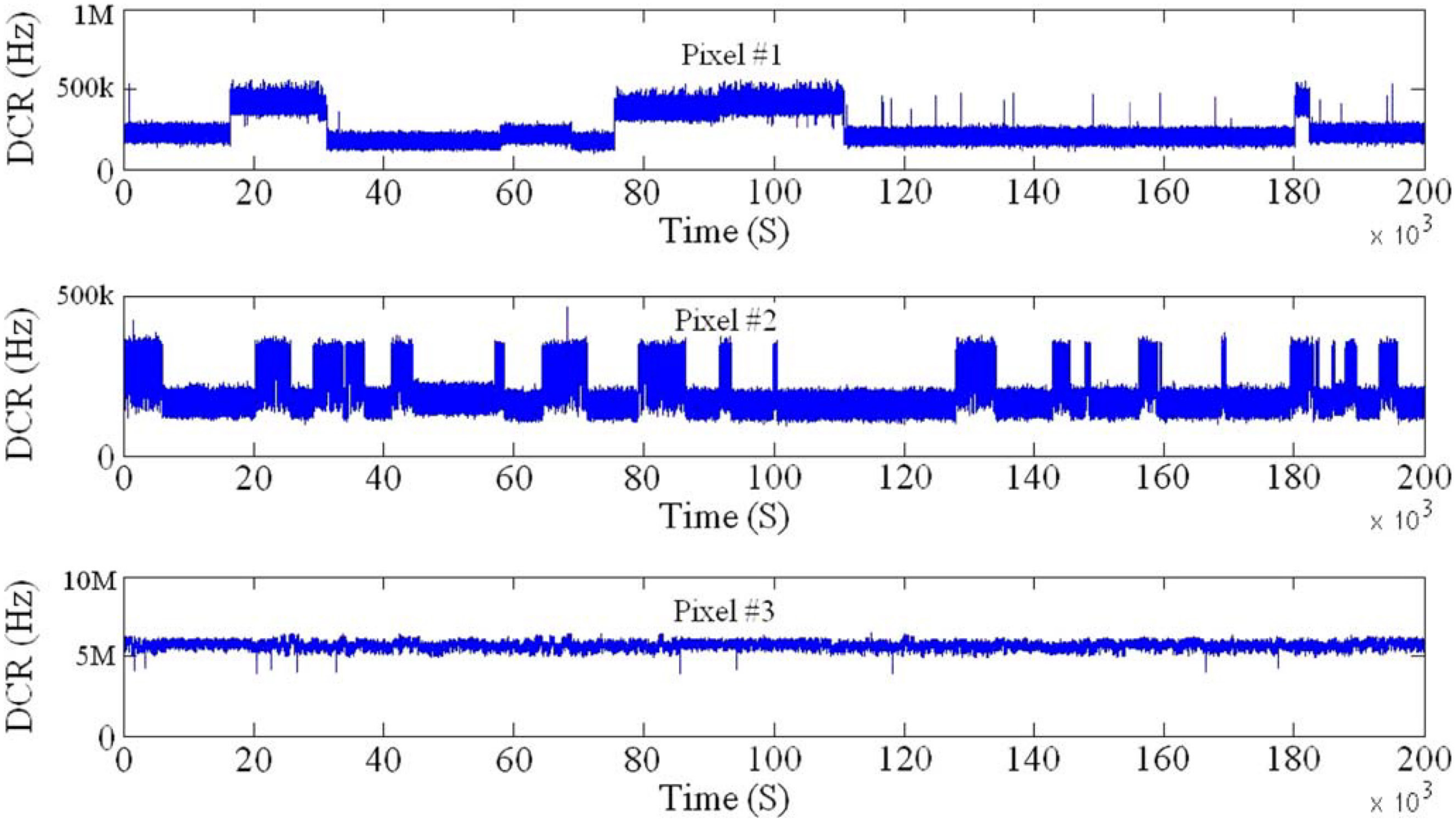
$$\text{Cross}-\text{covariance:} \qquad C_{XY}(t_1, t_2) = Cov\{X(t_1), Y(t_2)\}$$

NB: extended here to two RPs X and Y

$$\text{Cross}-\text{correlation:} \qquad K_{XY}(t_1, t_2) = E\{X(t_1) \cdot Y(t_2)\}$$

H. Bilgekul, Slides for the course "EE-461 Communication System II", EMU   F. Farahmand, Slides for the course "CES 540 Digital Communication", Ch. 6, SSU

9.1.1 Random Process (contd.) – Example, non-stationary



M. A. Karami et al., *Random Telegraph Signal in Single-Photon Avalanche Diodes*, International Image Sensor Workshop, Bergen, 2009

9.1.2 Stationary Random Process

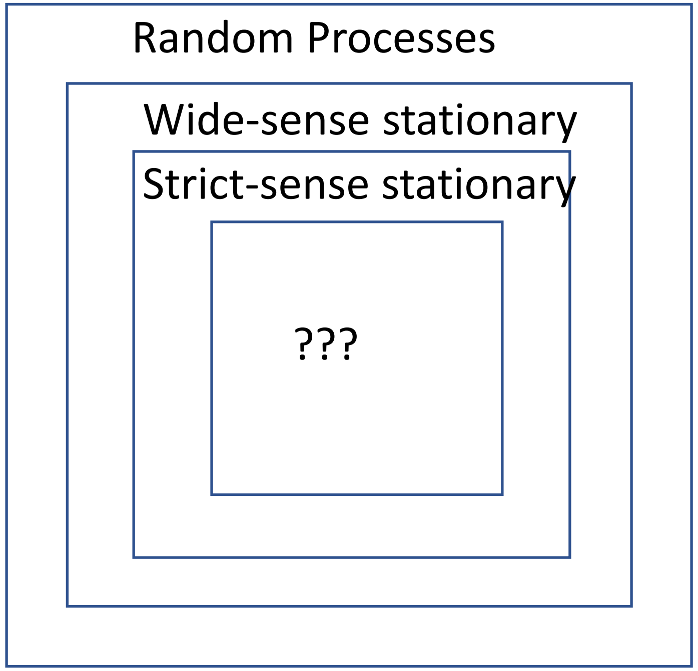
- We characterize the RP on how their statistical properties change in time. If they do not change, we call the RP stationary. Hence:

$$f_X(x,t) = f_X(x)$$

Ensemble averages

$$m_X(t) = \overline{X(t)} = E\{X(t)\} = \int_{-\infty}^{\infty} x\, f_X(x,t)\, dx = \mu_X$$

$$Var\{X(t)\} = E\left\{\left(X(t) - m_X(t)\right)^2\right\} = \int_{-\infty}^{\infty} \left(x - m_X(t)\right)^2 f_X(x,t)\, dx = \sigma^2$$



- Weaker form: in Wide-Sense Stationary RPs, in addition to a constant mean, the autocorrelation function only depends on the time difference, but not on the absolute position in time:

$$K_{XX}(t, t+\tau) = K_{XX}(\tau) \quad (\text{or equivalently } K_{XX}(t_1, t_2) = K_{XX}(t_2 - t_1))$$

Ex **WSS random process does not drift with time**

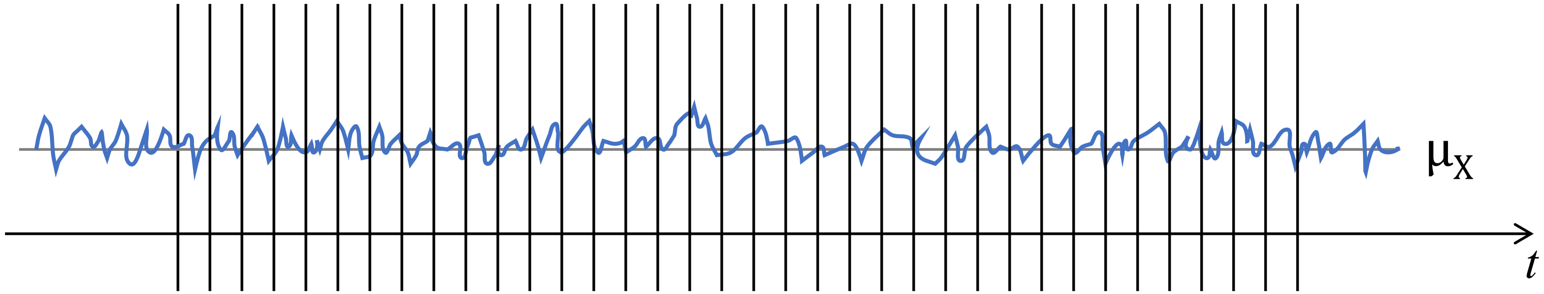
H. Bilgekul, Slides for the course "EE-461 Communication System II", EMU
F. Farahmand, Slides for the course "CES 540 Digital Communication", Ch. 6, SSU

9.1.2 Stationary Random Process – Example

$$f_X(x,t) = f_X(x)$$

$$m_X(t) = \overline{X(t)} = \mu_X$$

$$Var\{X(t)\} = \sigma^2$$



E. Charbon, "Image Sensors – ET 4390 Course Slides", Delft 2016

9.1.2 Stationary Random Process (contd.)

- For a Wide-sense Stationary Random Process $X(t)$, the autocorrelation function has the following properties:

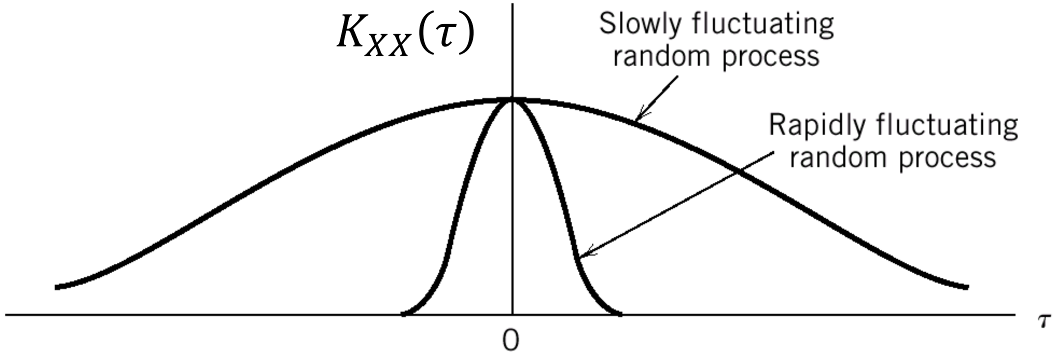
1. $K_{XX}(t_1, t_1) = K_{XX}(t_2, t_2) = K_{XX}(0) = E\{X^2(t)\} = \overline{X^2(t)} \geq 0$ ($\Longrightarrow K_{XX}\ (0)$ = *total power* of random signal $X(t)$, does not change in time)

2. $K_{XX}(\tau) = K_{XX}(-\tau)$

3. $\lim_{|\tau|\to\infty} K_{XX}(\tau) = \lim_{|\tau|\to\infty} E\{X(t) \cdot X(t+\tau)\} =$

   $= E\{X(t)\}\, E\{X(t+\tau)\} = \overline{X(t)}^{\,2}$ (example: *average or DC power* of random signal $X(t)$)

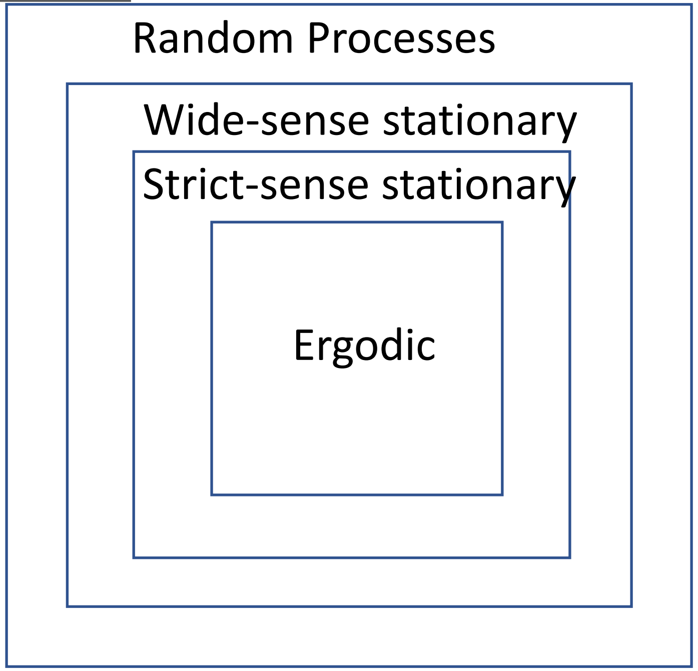
4. $|K_{XX}(\tau)| \leq |K_{XX}(0)| \qquad \text{for all } \tau$



F. Farahmand, Slides for the course “CES 540 Digital Communication”, Ch. 6, SSU

H. Bilgekul, Slides for the course “EE-461 Communication System II”, EMU

9.1.3 Ergodicity



- "A Random Process is ergodic if any sample function of the process takes all possible values in time with the same relative frequency that an ensemble will take at any given instant". Basically, its statistical properties can be deduced from a single, sufficiently long, random sample. [Wikipedia] Hence:

$$\overline{X(t)} = E\{X(t)\} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt = \langle X(t)\rangle$$

Ensemble function

Time average

$$K_{XX}(\tau) = E\{X(t)\cdot X(t+\tau)\} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt = \mathcal{K}_{XX}(\tau)$$

where $\langle X(t)\rangle$ is the time-average mean of the RP $X(t)$ and $\mathcal{K}_{XX}(\tau)$ is the time-average autocorrelation function.

"The ergodic hypothesis is that personal experience over time of a single individual reflects the current statistics of the general population."

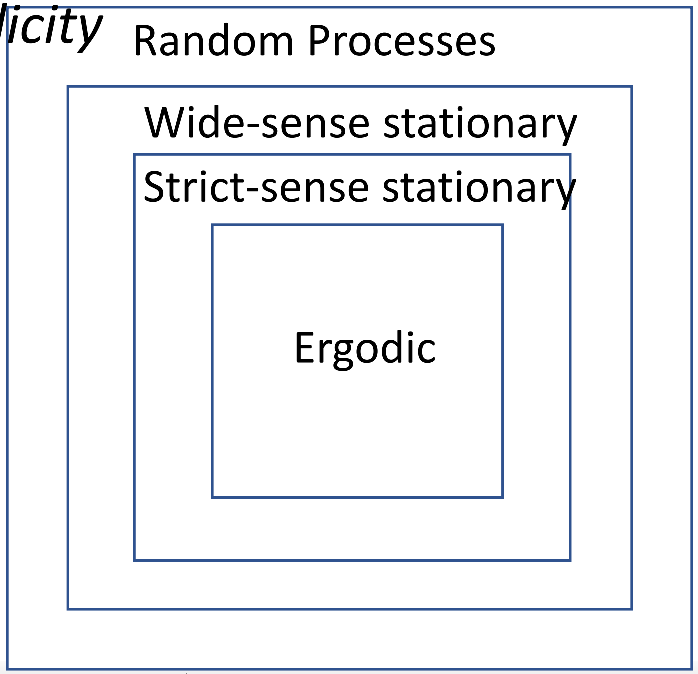
(I. Vardi)

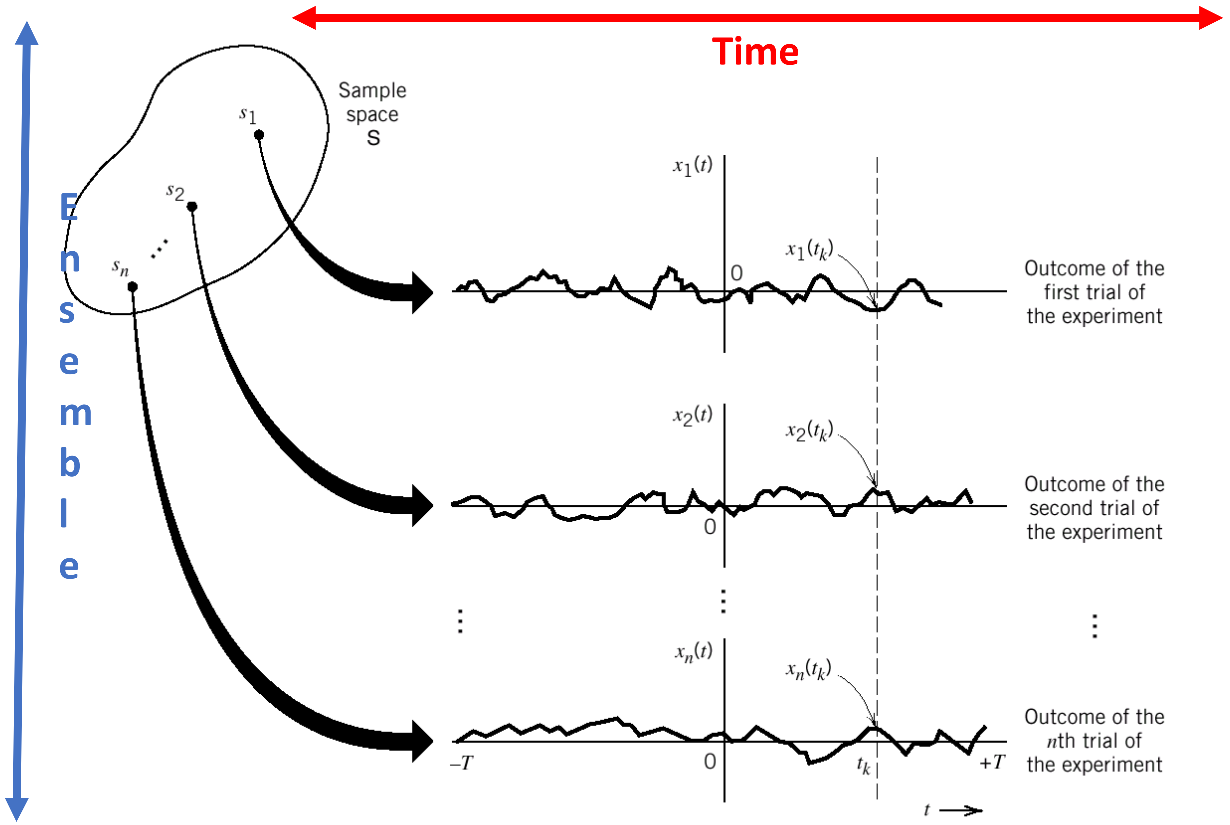
H. Bilgekul, Slides for the course "EE-461 Communication System II", EMU

F. Farahmand, Slides for the course "CES 540 Digital Communication", Ch. 6, SSU

Take-home Messages/W9-2

- *Random Process:*
  - Definition, Ensemble vs. Time
  - CDF, PDF, Moments, Autocorrelation
  - Wide-sense & strict-sense stationary
  - *Ergodicity*





Outline

9.2.1 Law of Large Numbers

- Law of large numbers: describes the behavior of the sample mean of i.i.d. random variables as the sample size grows
- Assume i.i.d. $X_1, X_2, X_3, \ldots$ with finite mean μ and finite variance σ^2

NB: i.i.d. = independent and identically distributed Random Variables, have the same PDF and are all mutually independent

$$\text{Sample Mean:} \qquad \overline{X_n} = \frac{X_1 + \cdots X_n}{n}$$

- $\overline{X_n}$ itself a random variable with

$$\text{Mean:} \qquad E\{\overline{X_n}\} = \frac{1}{n} E\{X_1 + \cdots X_n\} = \frac{1}{n}(E\{X_1\} + \cdots + E\{X_n\}) = \mu$$

$$\text{Variance:} \qquad \mathrm{Var}\{\overline{X_n}\} = \frac{1}{n^2} Var\{X_1 + \cdots X_n\}$$

$$= \frac{1}{n^2}(Var\{X_1\} + \cdots + Var\{X_n\}) = \frac{\sigma^2}{n}$$

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 10.2

9.2.1 Law of Large Numbers (contd.)

- Law of Large Numbers: as n grows, the sample mean $\overline{X_n}$ converges to the true mean μ
- Essential for simulations, statistics, etc. – implicitely used when we use:

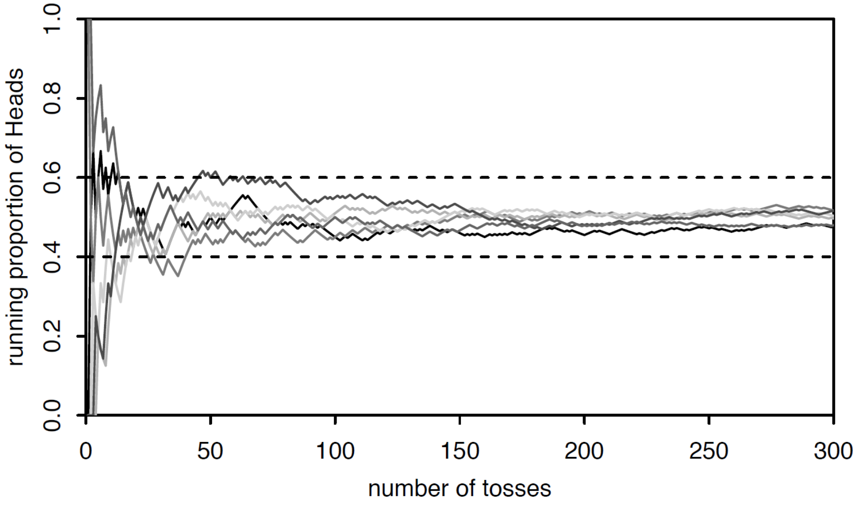
1) the proportion of times that something happened as an approximation to its probability,

2) the average value in the replications of some quantity to approximate its theoretical average.

Example: *improvement in LiDAR ranging precision…*

…when accumulating timing measurements, as 1/n

Q



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 10.2

9.2.2 Central Limit Theorem

- Law of Large Numbers: as n grows, the sample mean $\overline{X_n}$ converges to the true mean μ

But with which distribution? **Q**

Sum of a large number of i.i.d. random variables has an approximately Gaussian (normal distribution),

- regardless of the distribution of the individual RVs (could be anything!)
- very weak assumptions.

$$\text{As } n \to \infty, \qquad \sqrt{n}\left(\frac{\overline{X_n} - \mu}{\sigma}\right) \sim \mathcal{N}(0,1)$$

(i.e. the CDF of the l.h.s. approaches Φ, the CDF of the standard Gaussian distribution)

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 10.3

9.2.2 Central Limit Theorem (contd.)

- In other words: start with independent RVs from almost any distribution, discrete or continuous,
  -> add them up
  -> distribution of the resulting RV has a Gaussian shape!
- The CLT is an asymptotic result. Approximation: for large n

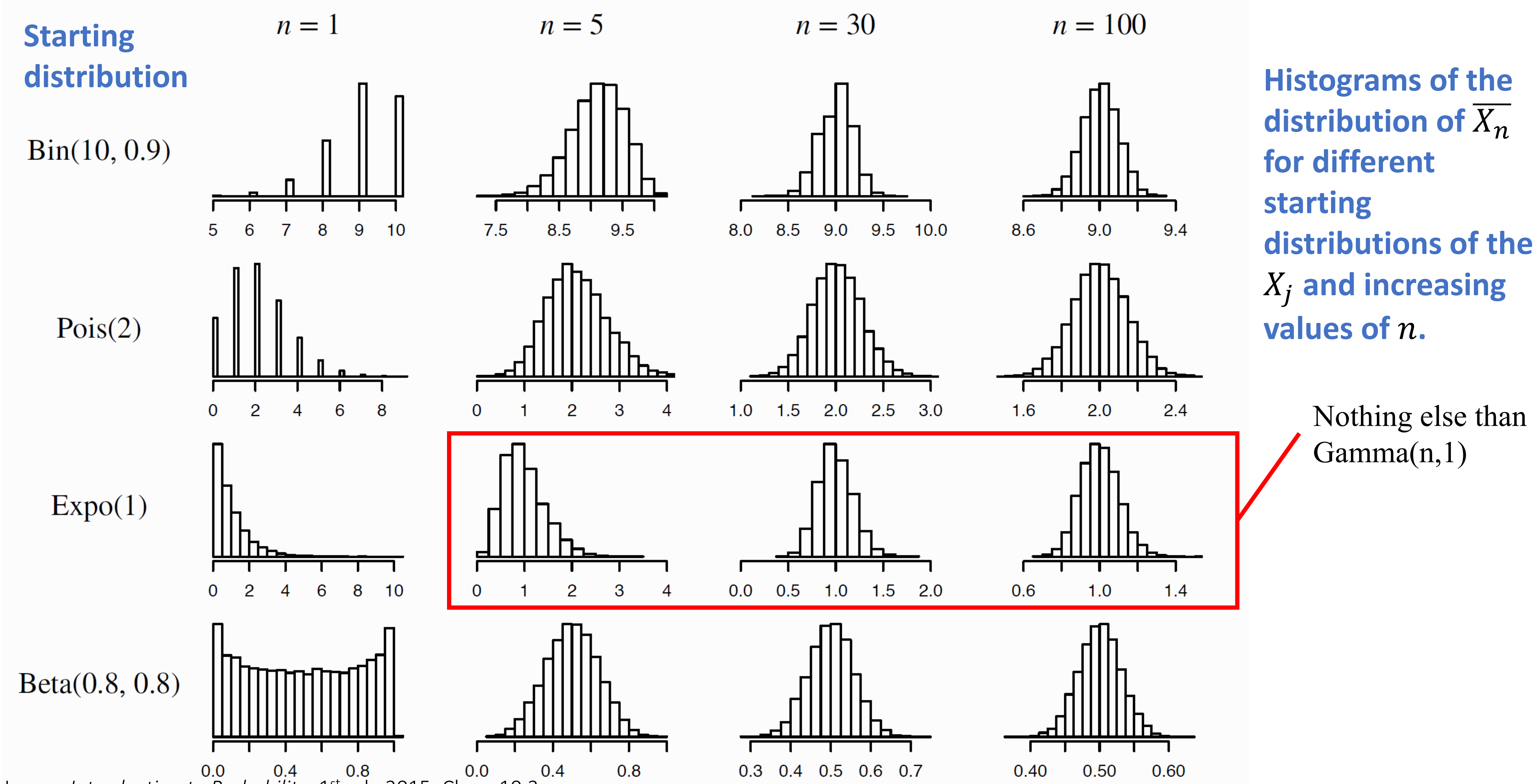
$$\sqrt{n}\left(\frac{\overline{X_n}-\mu}{\sigma}\right) \to \mathcal{N}(0,1)$$

- NB: The distribution of the X_j is still relevant, e.g. if highly skewed or multimodal, n might need to be very large before the Gaussian approximation becomes accurate.

- Conversely, if the X_j are already i.i.d. Normals (Gaussian), the distribution of $\overline{X_n}$ is exactly $\mathcal{N}\left(\mu, {}^{\sigma^2}/_{n}\right)$ for all *n*.

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 10.3

9.2.2 Central Limit Theorem - Example



Histograms of the distribution of $\overline{X_n}$ for different starting distributions of the X_j and increasing values of n.

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 10.3

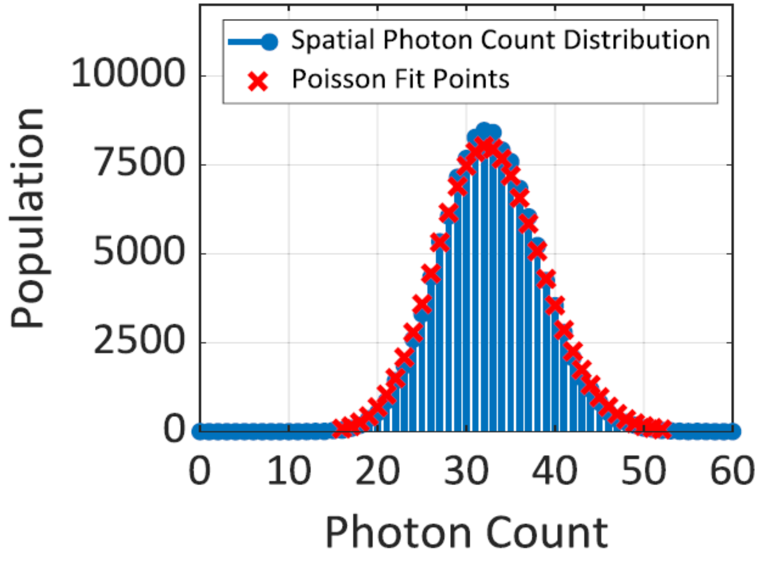
9.2.2 Central Limit Theorem - Example

- Poisson convergence to Gaussian: if

$$Y \sim Pois(n)$$

we can consider Y as a sum of n i.i.d. Pois(1) RVs.

$$\text{For large } n\text{: } Y \rightarrow \mathcal{N}(n, n)$$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 10.3

Outline

9.3 Elements of Estimation Theory

- Estimation theory has the purpose to solve one problem: given a set of data

$$\{x_1, x_2, \dots, x_{N-1}\}$$

which depends on an unknown parameter vector θ, determine an estimator

$$\hat{\theta} = g(x_1, x_2, \dots, x_{N-1})$$

where g is some function.

- In other words, how do we use collected data to estimate unknown parameters of a distribution?

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 6.3

9.3 Elements of Estimation Theory (contd.)

- In general, if we assume that θ is deterministic, we will have a classical estimation problem. It can be solved in the following ways, and many more:

1. Least Squares Estimator (LSE)
2. Minimum Variance Unbiased Estimator (MVU)
3. Maximum Likelihood Estimator (MLE) *Ex*
4. Best Linear Unbiased Estimator (BLUE)
5. …

9.3.1 Elements of Estimation Theory – Simple Mean Example

- Simple example: estimate the mean of a sample of i.i.d. RVs $X_1, X_2, X_3, \ldots, X_n$

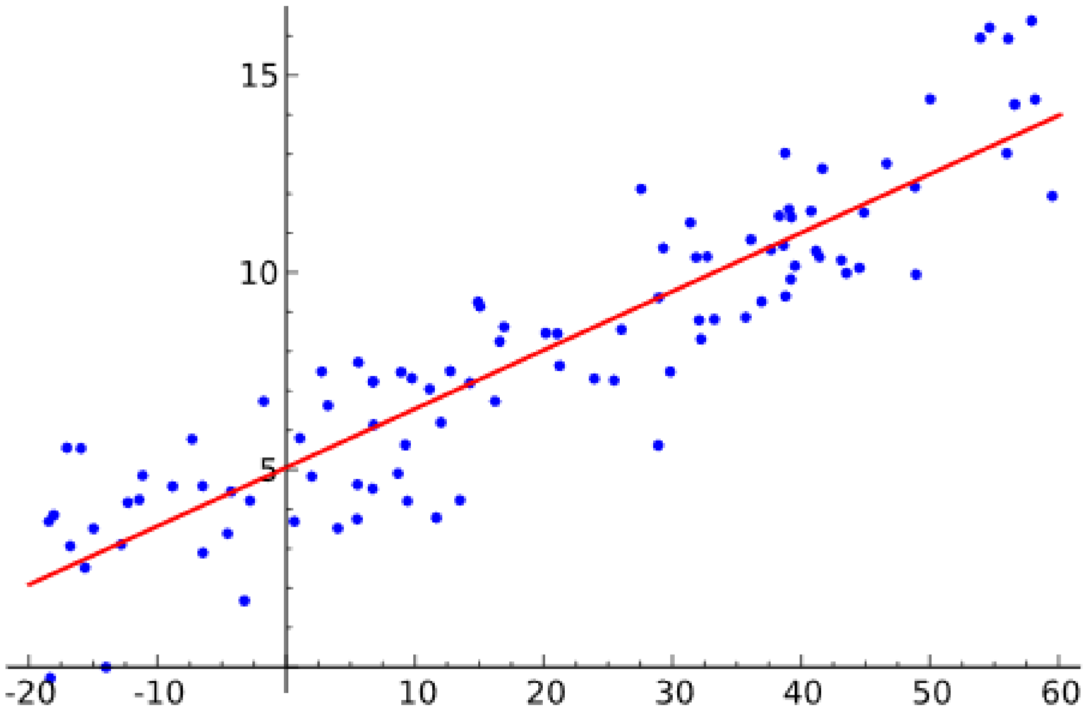
$$\text{Sample Mean: } \overline{X_n} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

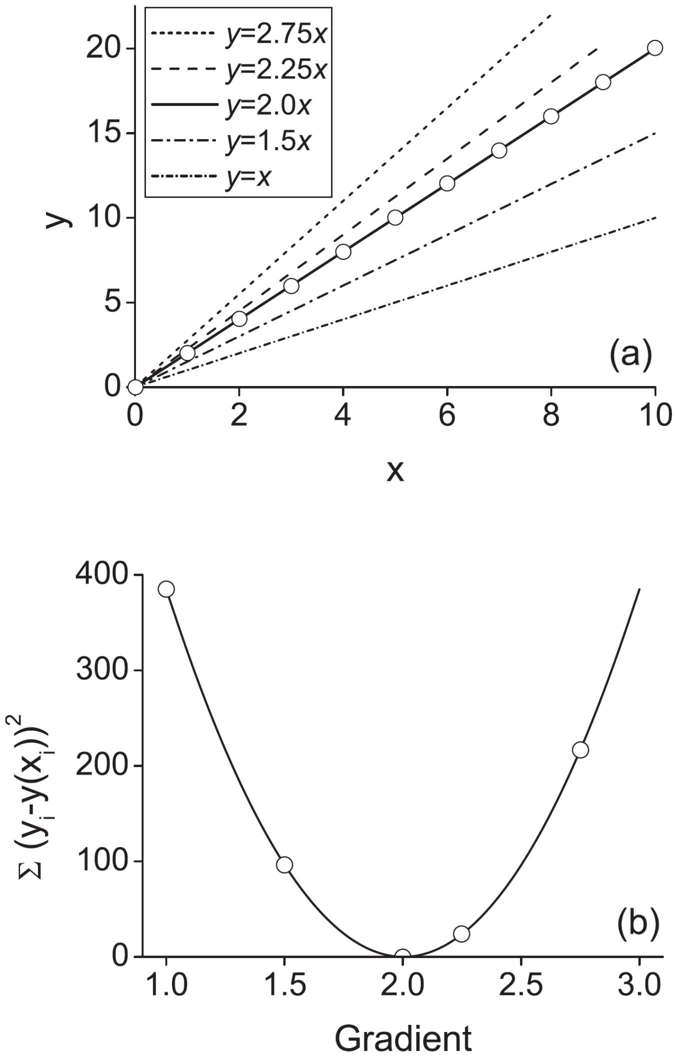
is an estimate of the *population mean* or *true mean,* $E\{X_j\}$ = the mean of the distribution from which the X_j were drawn.

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 6.3

9.3.2 Elements of Estimation Theory – LSE Example

Least Squares Estimator (LSE)



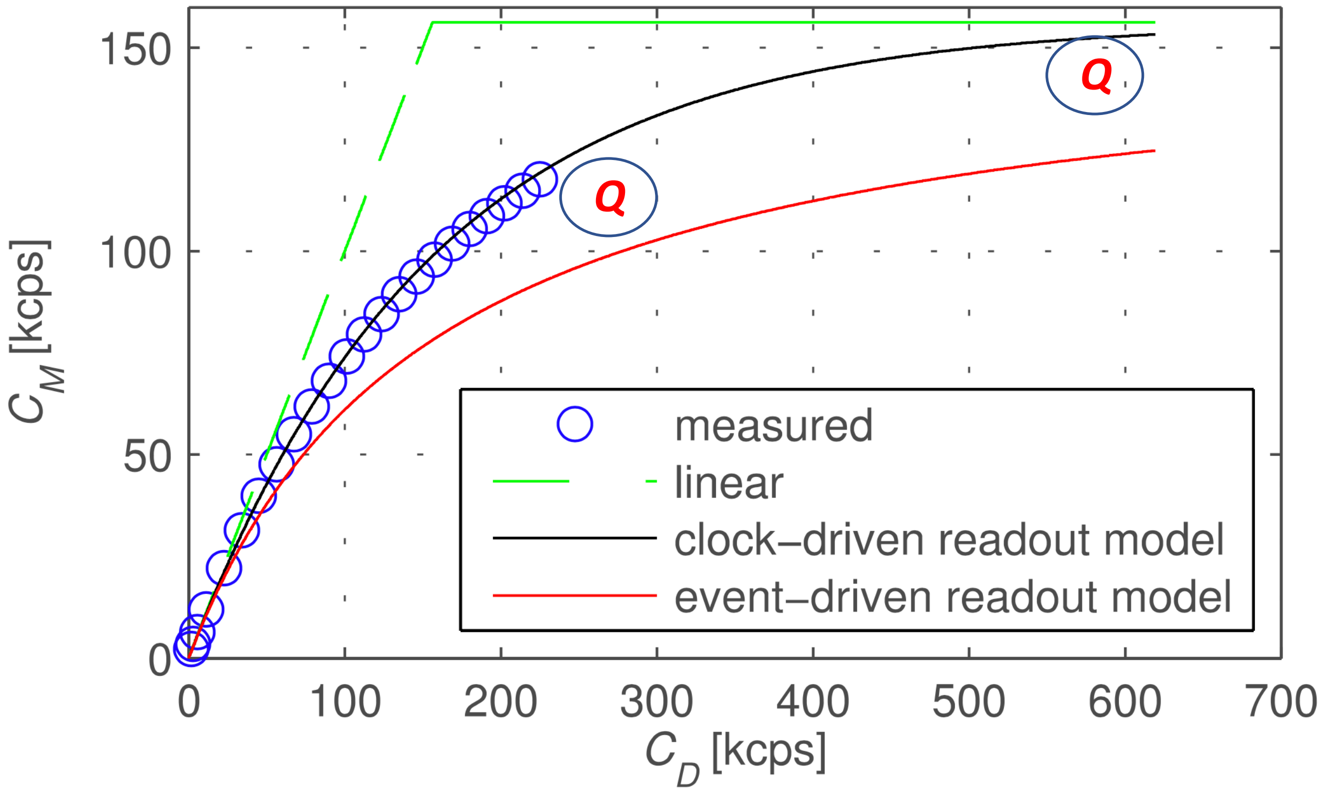


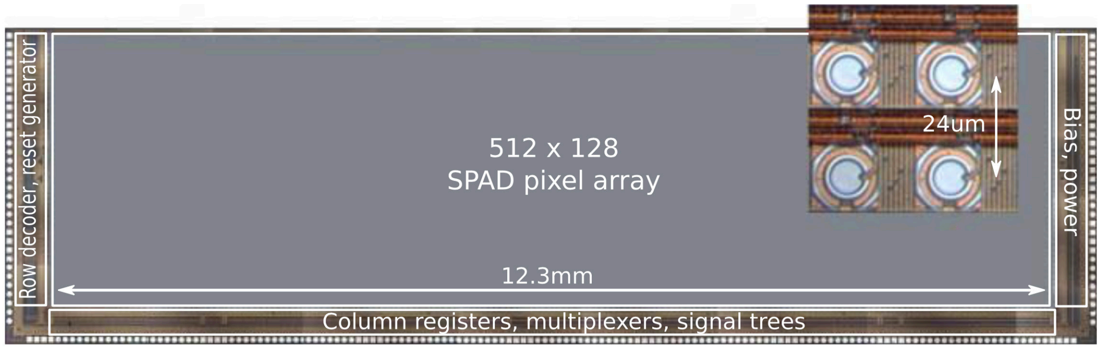
📖 https://commons.wikimedia.org/wiki/File:Linear_regression.svg

📖 I.G. Hughes, T.P.A. Hase, *Measurements and their Uncertainties*, 1st ed., 2010, Chap. 6.3

9.3.3 Elements of Estimation Theory – MLE Example

Maximum Likelihood Estimator (MLE) to correct for the exponential count loss in binary, clock-driven SPAD imagers





C_M: Measured count rate (externally)

C_D: Detected count rate (internally)

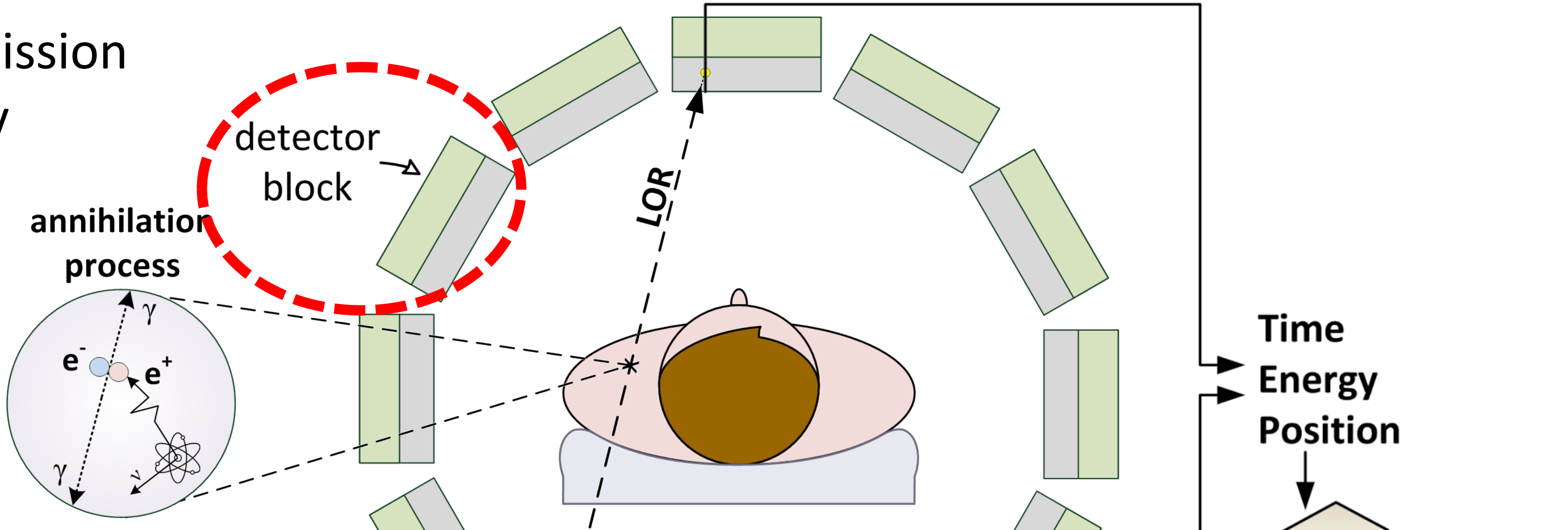
$$E[C_M] = \frac{1 - e^{-C_D \times T_{\text{readout}}}}{T_{\text{readout}}}$$

$$E[C_D] = \frac{-\ln(1 - C_M \times T_{\text{readout}})}{T_{\text{readout}}}$$

I. M. Antolovic et al, *Nonuniformity Analysis of a 65-kpixel CMOS SPAD Imager*, IEEE Trans. on Electron Devices 63, 2016

9.3.4 Elements of Estimation Theory – BLUE Example

Positron Emission Tomography Basics

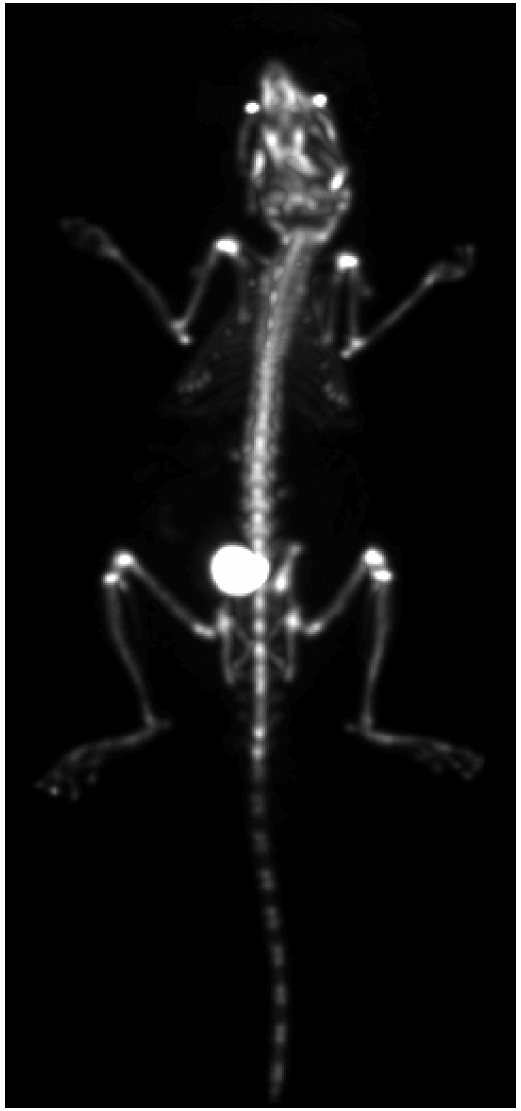


GE Discovery IQ, Nov 2016

L. Braga *et al.*, ISSCC, 2013

9.3.4 Elements of Estimation Theory – BLUE Example

Positron Emission Tomography Reconstruction Example



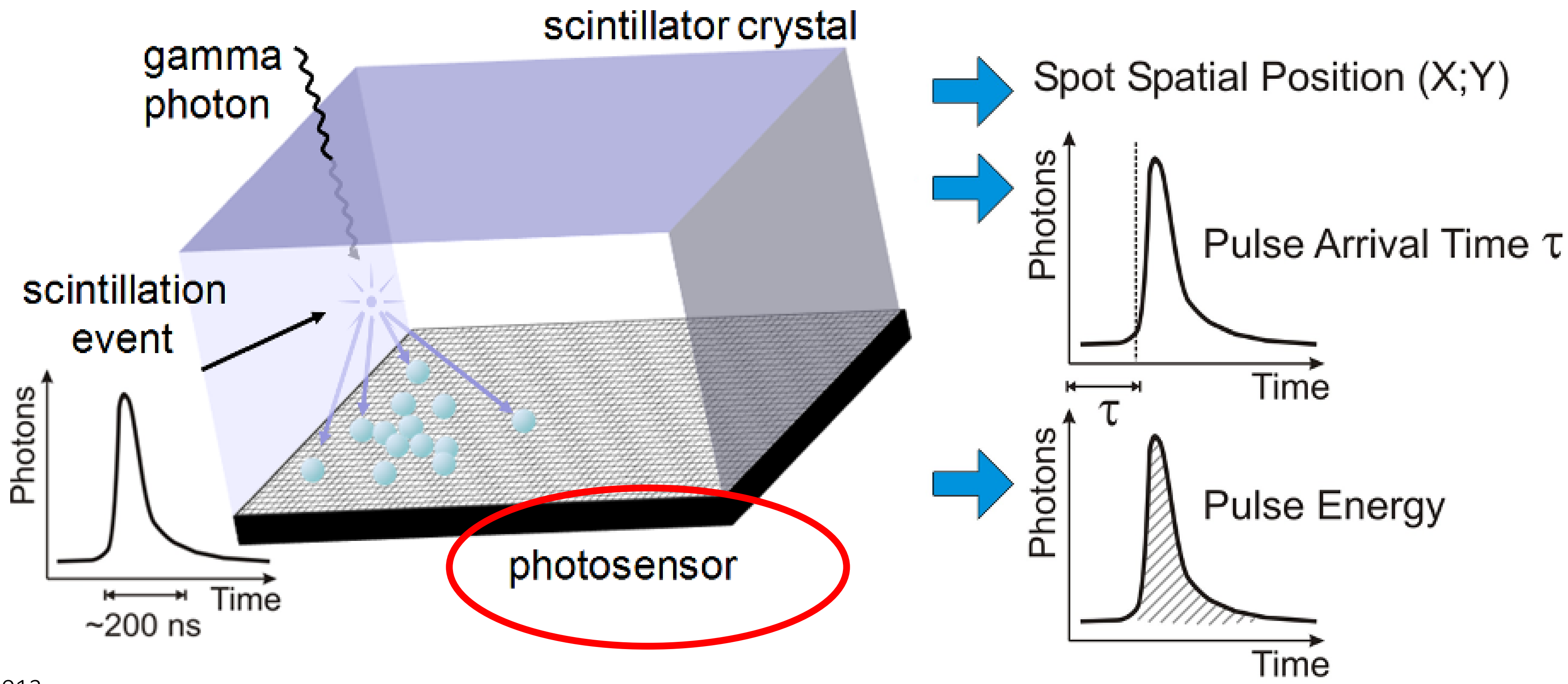
G. Nemeth, Mediso, Delft WS 2010

9.3.4 Elements of Estimation Theory – BLUE Example

Positron Emission Tomography Building Blocks & Main Variables

Problem: estimate the scintillation event time T_0 given a set of timing measurements t_q

Aim: obtain estimator with lowest variance (best timing precision)



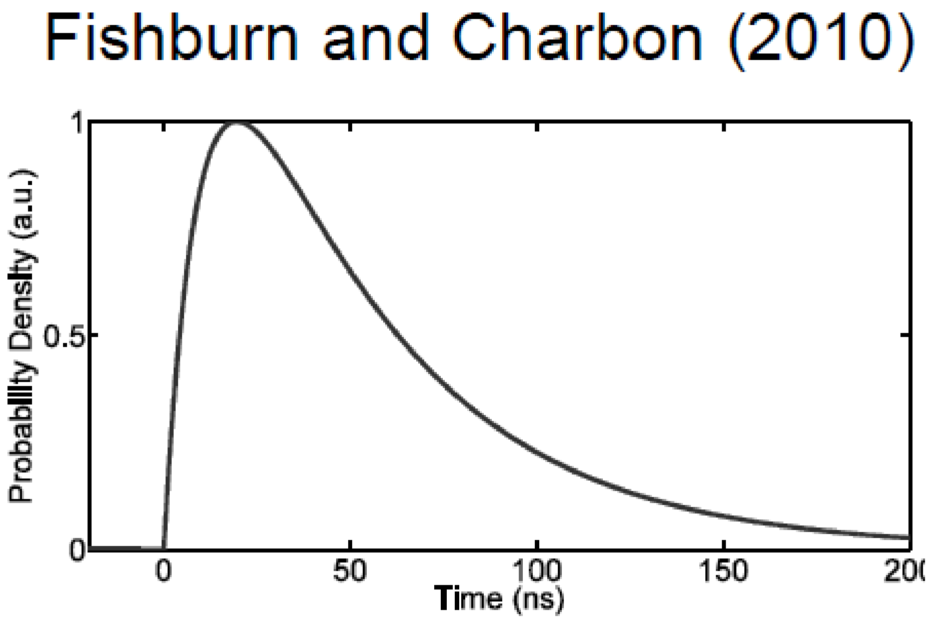
R. Walker *et al.*, IISW, 2013

Typ some 10^4 photons/scintillation, few 10^3 detected

9.3.4 Elements of Estimation Theory – BLUE Example

LYSO: τ_{rise}~70ps, τ_{fall}~30ns

- ❑ The scintillation follows a double exponential decay.
- ❑ The transit time spread is modelled as additive noise.
- ❑ The best timing performance might not be obtained with the first photoelectron



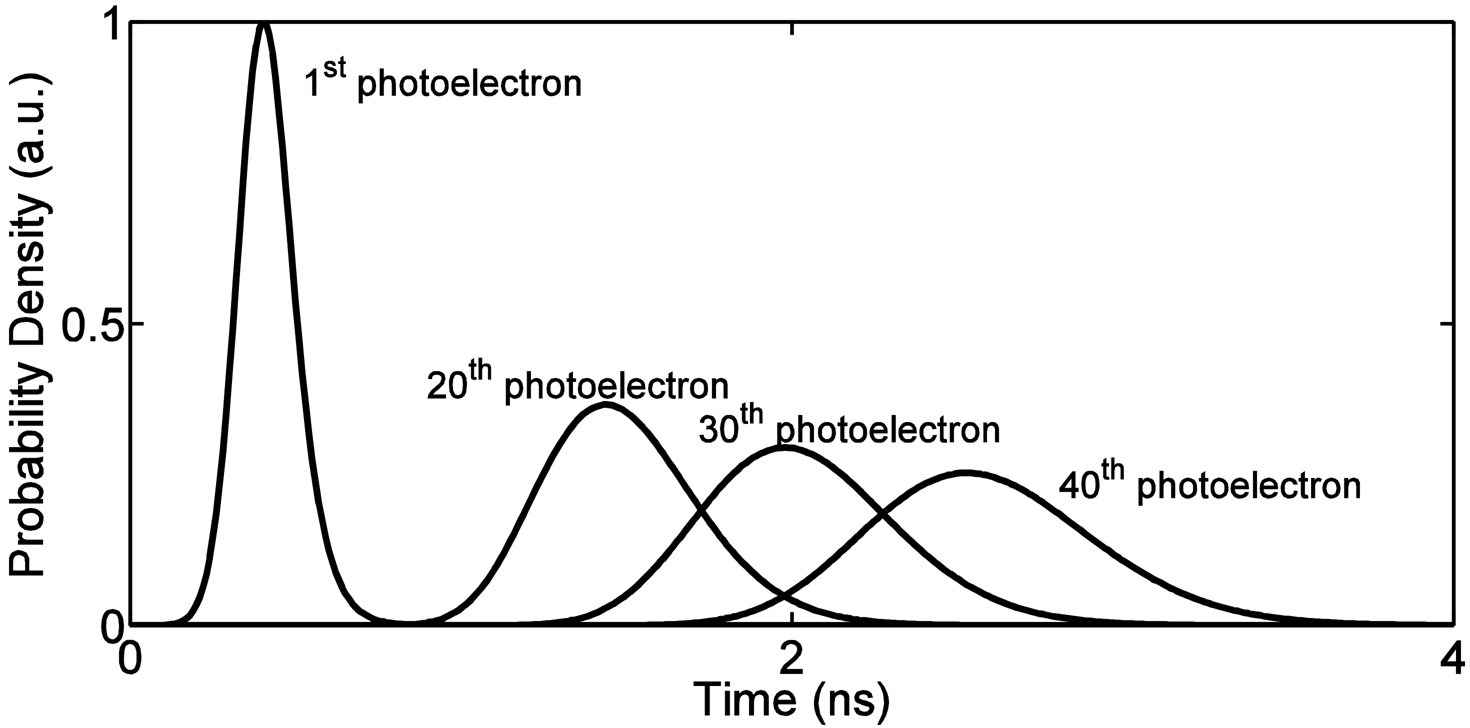
M. Fishburn, E. Charbon, NSS-MIC, 2012

M. Fishburn, E. Charbon, IEEE TNS(57), 2010

E. Venialgo *et al.*, NSS-MIC, 2015

9.3.4 Elements of Estimation Theory – BLUE Example

Order Statistics



$f(t), F(t)$ = scintillation PDF/CDF (previous slide)

$$p_q(t) = \frac{R!}{(q-1)!(R-q)!}[1 - F(t)]^{(R-q)}[F(t)^{(q-1)}]f(t),$$

PDF of the q^{th} photoelectron's time-of-registration.

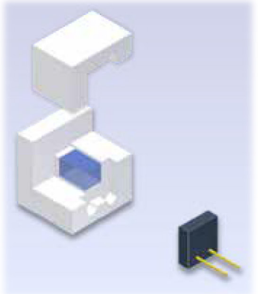
Order statistics implies correlation between timestamps

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Theorem 8.6.4

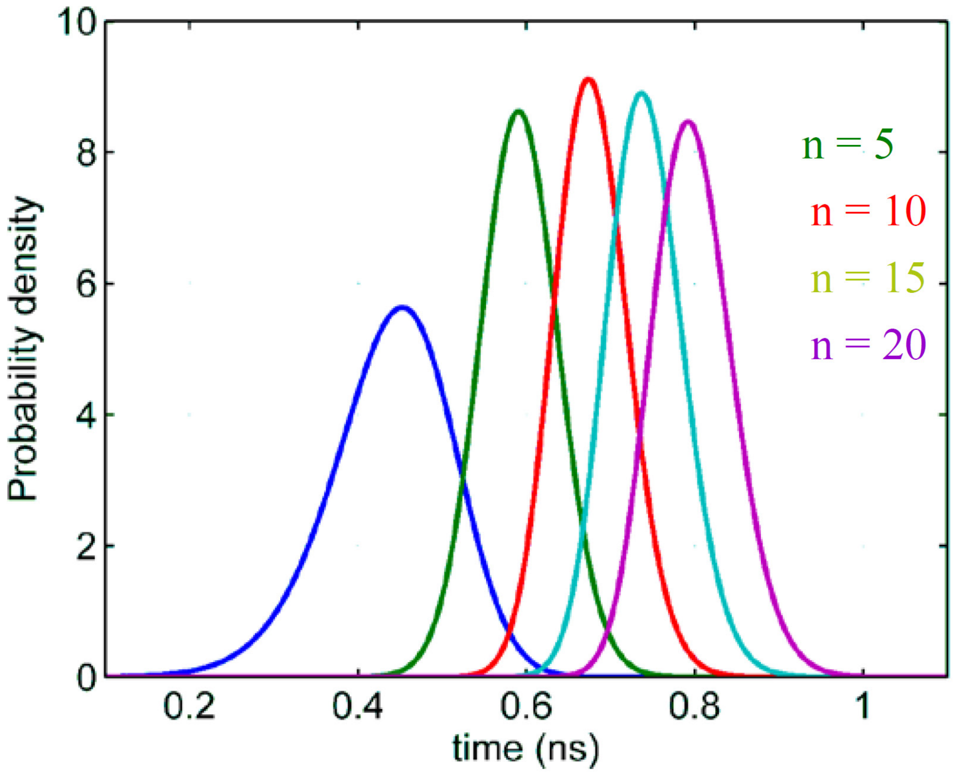
E. Venialgo, E. Charbon *et al.*, PMB 2015

9.3.4 Elements of Estimation Theory – BLUE Example

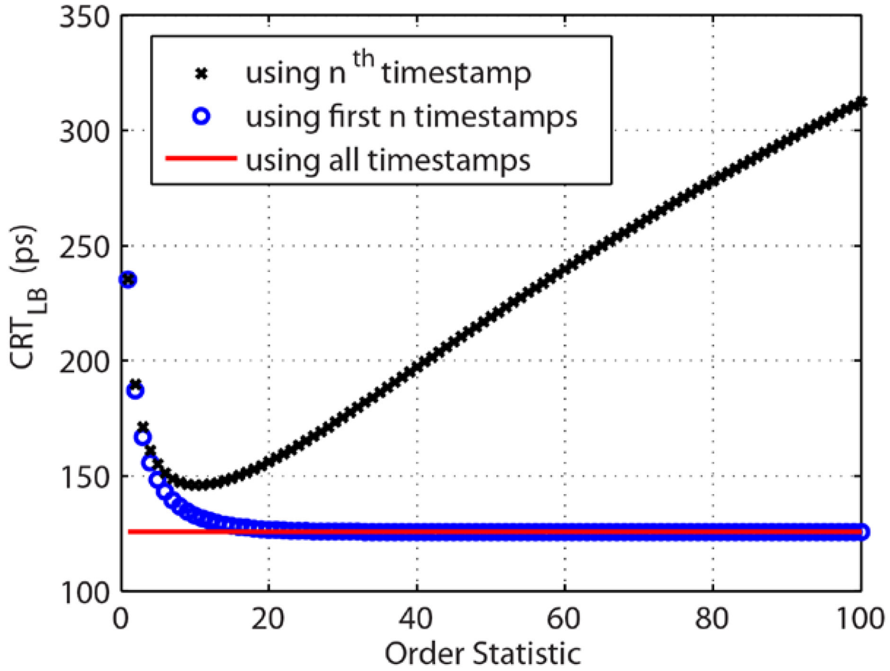
Lower bound for LYSO:Ce



Timestamp for the n^{th} detected scintillation photon



Exemplary probability density functions for the n^{th} order statistic for LYSO:Ce on MPPC-S10362-33-050C



Lower bound on the CRT for LYSO:Ce on MPPC-S10362-33-050C, using the n^{th}, the first n, or all detected photons ("order statistics") for timing

Parameters:

$\tau_r = 90$ ps

$\tau_d = 44$ ns

$\sigma = 120$ ps

$N_{det} = 4700$

D. Schaart, ANSRI 2016, 2016, Dublin, Ireland

9.3.4 Elements of Estimation Theory – BLUE Example

$$\hat{T}_0^{(p)} = \sum_{q=1}^{Q} t_q w_q^{(p)}, \; p = 1, 2, 3.$$

← General estimator

(p is one of 3 possible estimators)

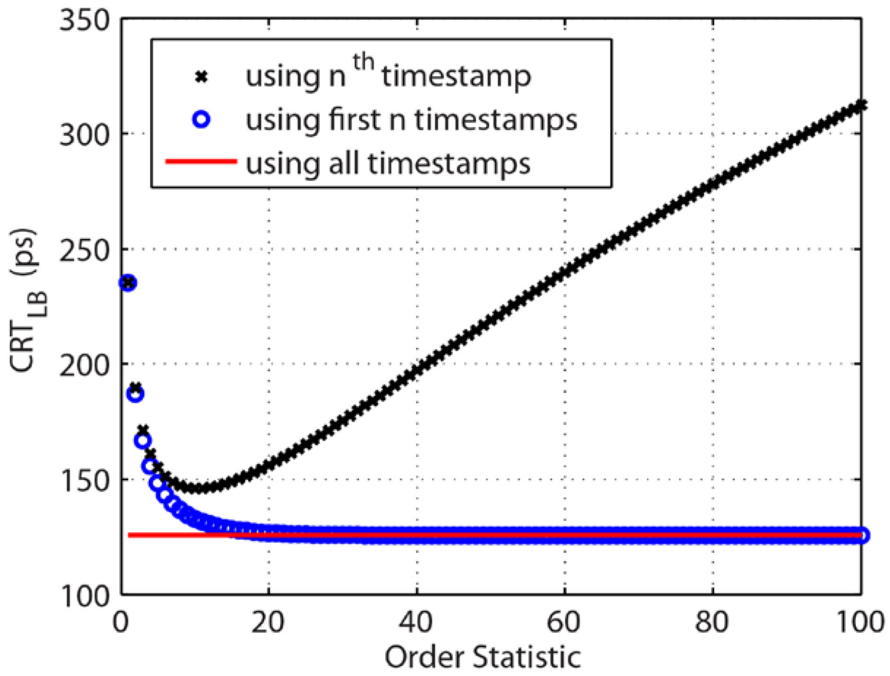
Q

A simple estimator approach:

$$w_q^{(1)} = \frac{1}{Q}, \; q = 1, \ldots, Q$$

← Simple mean coeffs.

($p = 1$ estimator)



D. Schaart, ANSRI 2016, 2016, Dublin, Ireland

E. Venialgo, E. Charbon *et al.*, PMB 2015

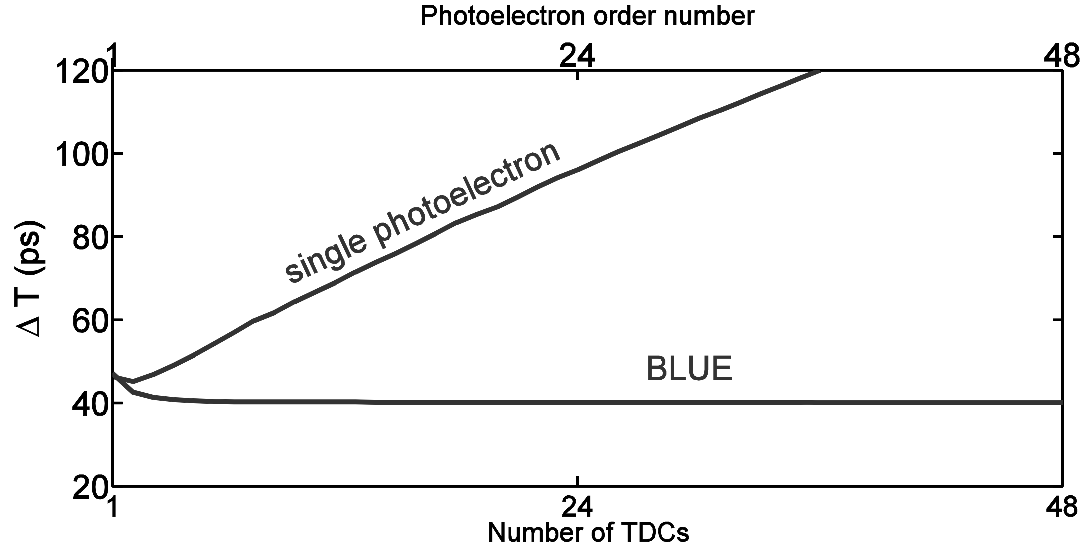
9.3.4 Elements of Estimation Theory – BLUE Example

- Assuming large number of measurements
- Other Estimator Approaches: **B**est **L**inear **U**nbiased **E**stimator (BLUE)

$$\hat{T}_0^{(p)} = \sum^{Q} t_q w_q^{(p)},\ p = 1, 2, 3. \longleftarrow$$ General estimator

$$w_q^{(3)} = \frac{C^{-1}d}{\|C^{-1/2}d\|_2^2}, \longleftarrow$$ BLUE coeffs. (correlation matrix)

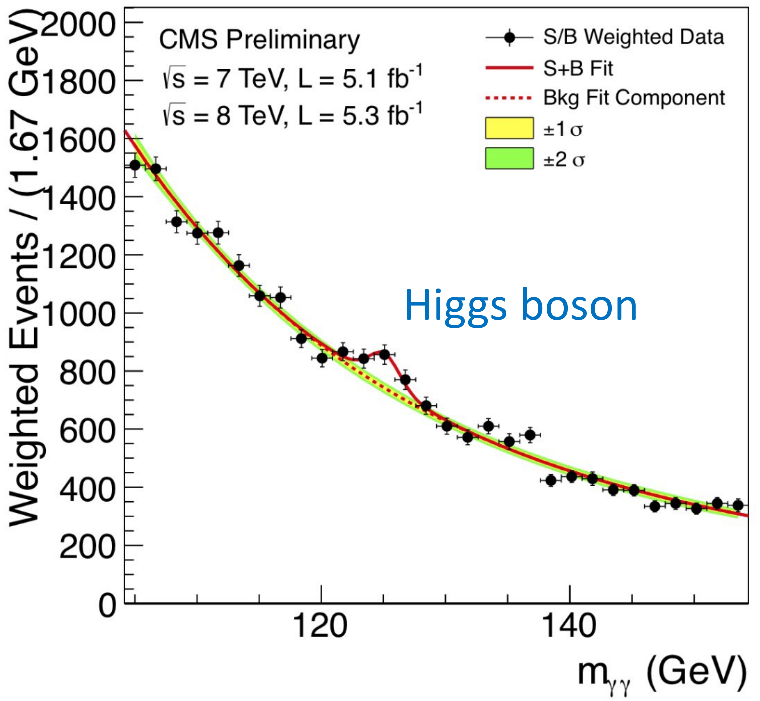
($p = 3$ estimator)



E. Venialgo, E. Charbon *et al.*, PMB 2015

Outline

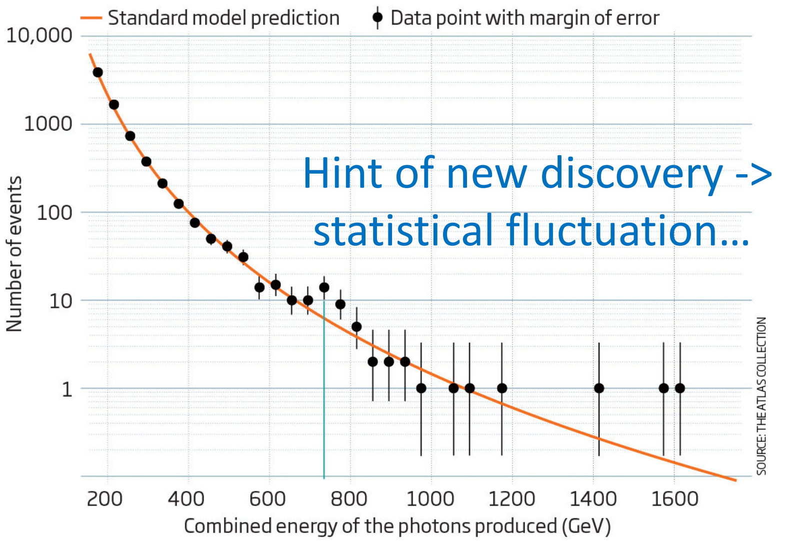
9.4 Error Analysis



- The aim of error analysis is to quantify and record the errors associated with the inevitable spread in a set of measurements.

- Confidence boundaries represent the quality of the approximation given by the uncertainty.

Example: the six-sigma method, 5 sigma limit (CERN) *Q*



- Uncertainties can be associated to random errors (hence influencing the variance of the measurement distribution) or to systematic errors (acting on the mean value of the measurement distribution).

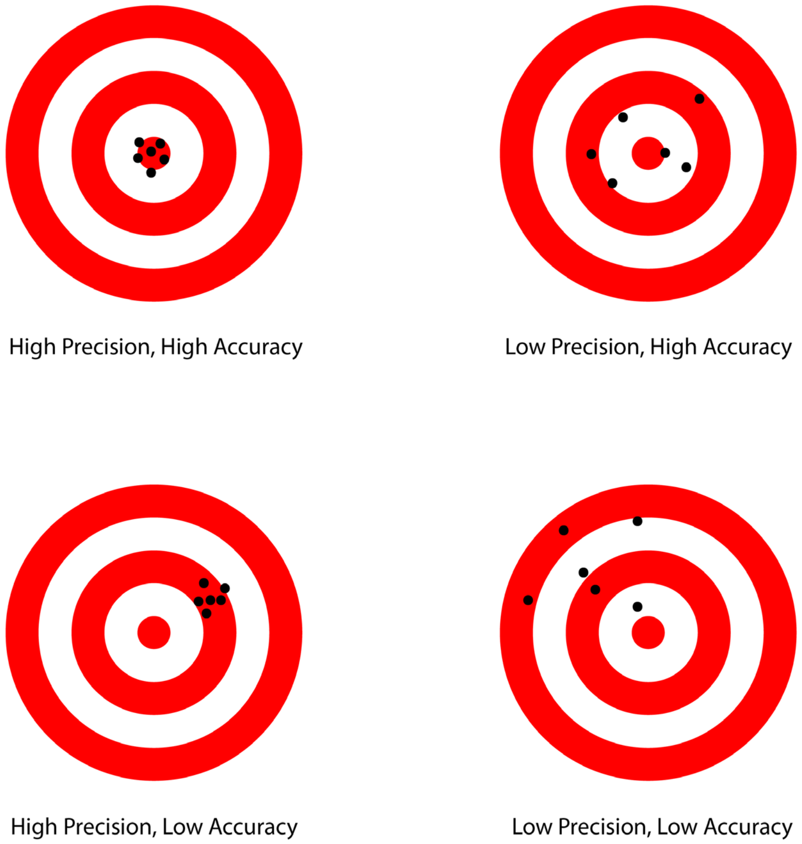
I.G. Hughes, T.P.A. Hase, *Measurements and their Uncertainties*, 1st ed., 2010, Chap. 1

https://universe-review.ca/R15-20-accelerators03.htm

9.4.1 Accuracy

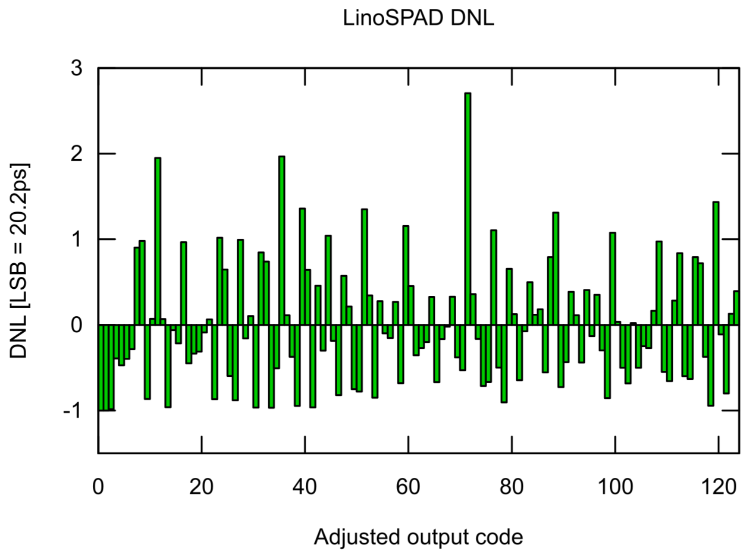
Accuracy -> mean

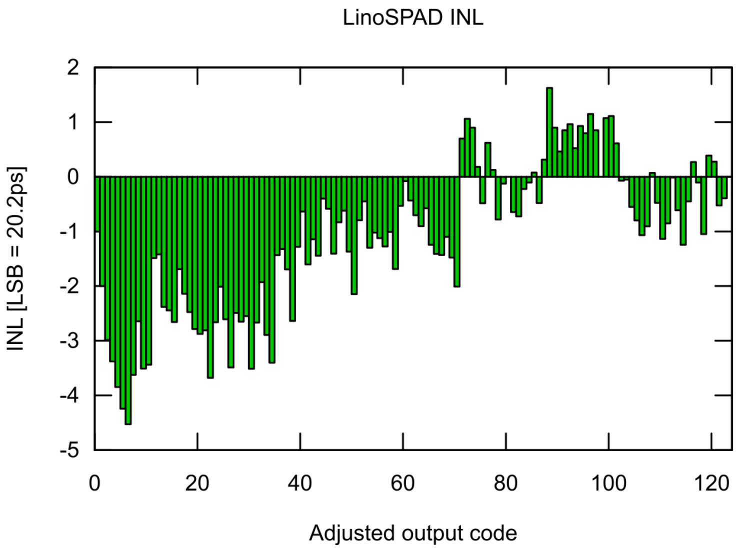
- The accuracy of a measurement gives a notion of the mean value of the set of measurements distribution with respect to the real value.
- An accurate measurements distribution will hence have a very small systematic error, but could be affected by a large spread in the data (high variance).
- Accuracy can be enhanced in the experimental real life by means of calibration techniques.



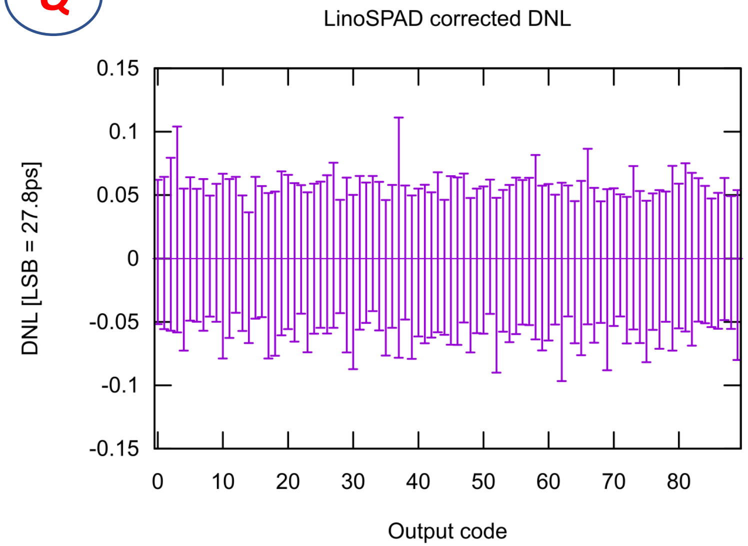
I.G. Hughes, T.P.A. Hase, *Measurements and their Uncertainties*, 1st ed., 2010, Chap. 1

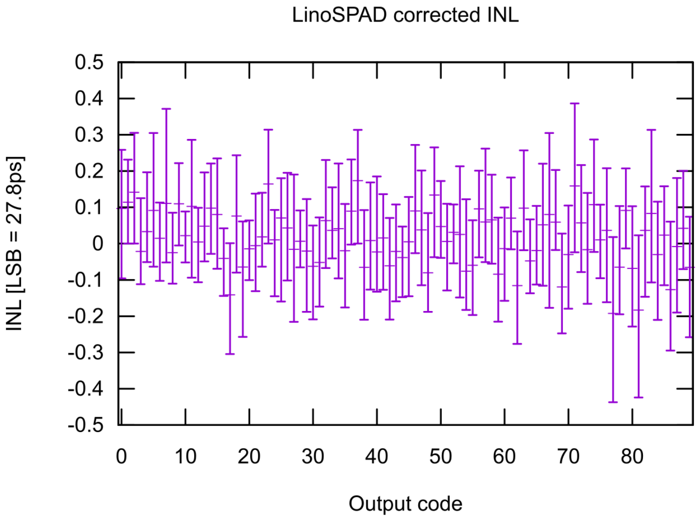
9.4.1 Accuracy – Example





Q





Calibration of a Time-to-Digital converter
S. Burri, EPFL, MDPI Instruments, 2018

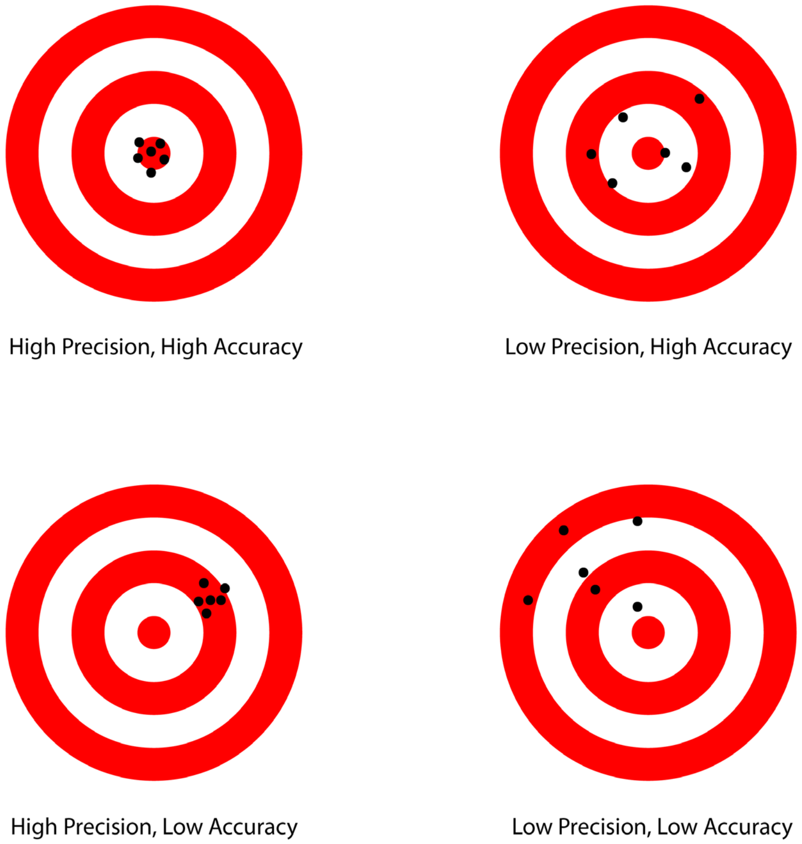
9.4.2 Precision

Precision -> spread (variance)

- The precision of a measurement gives information about the spread of the measured set of data collected by the measurement.

- A precise measurement distribution will have a low dispersion of data (hence a small variance), but it might have a mean value very distant from the real one.

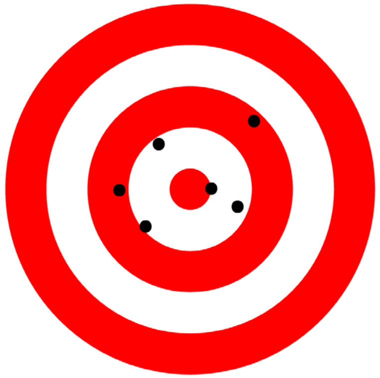
- In order to enhance precision, the most simple way is to increase the size of the sample data. In fact, as shown previously, for experimental data the variance decreases linearly with the number of samples collected.



I.G. Hughes, T.P.A. Hase, *Measurements and their Uncertainties*, 1st ed., 2010, Chap. 1

9.4.2 Accuracy vs. Precision

High Precision, High Accuracy



Low Precision, High Accuracy

High Precision, Low Accuracy

Low Precision, Low Accuracy

9.4.3 Resolution

- The resolution of a measurement is the smallest change in the underlying physical quantity that produces a response in the measurement. [Wikipedia]

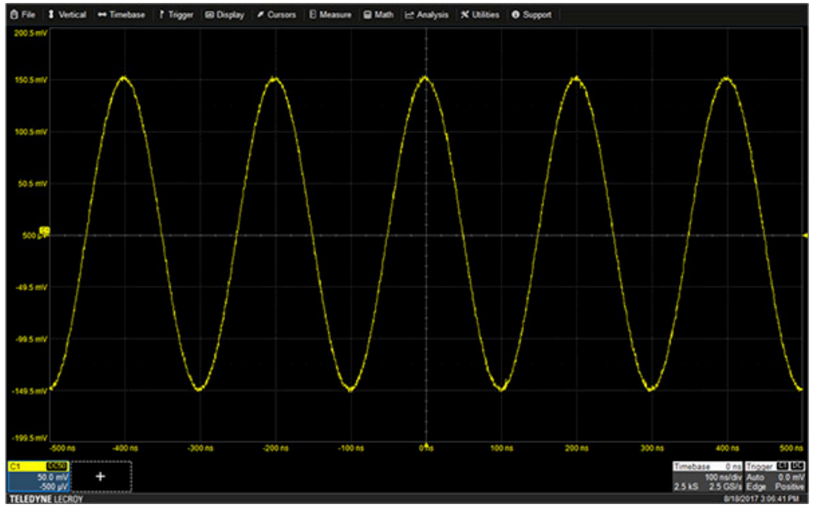
- In case of an ADC (analog-to-digital converter), the resolution is given by one bit.

Example: for an oscilloscope with an 8 bits ADC, set at 100 mV/div (i.e. for a total screen width of 800 mV), the resolution of each point collected is given by:

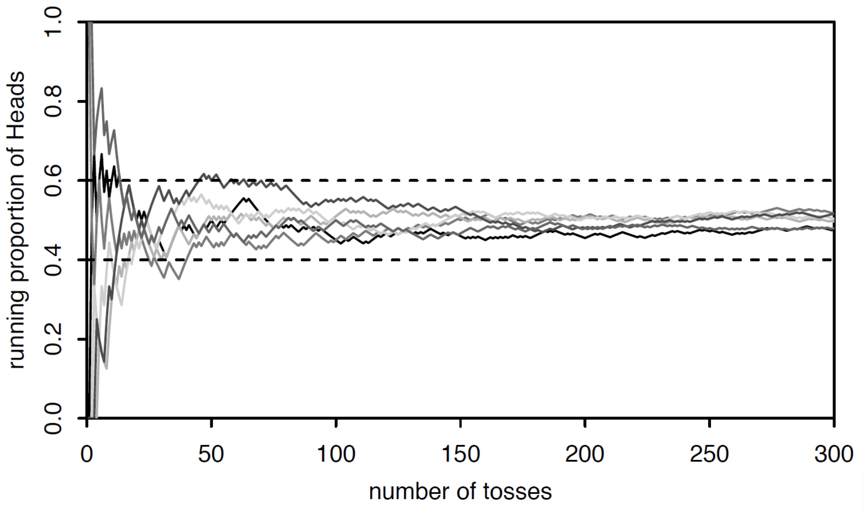
$$8 \text{ bits} = 2^8 \text{ different values} \rightarrow Res = \frac{800}{256}\, mV = 3.125\, mV$$

High resolution 10 Downing street.

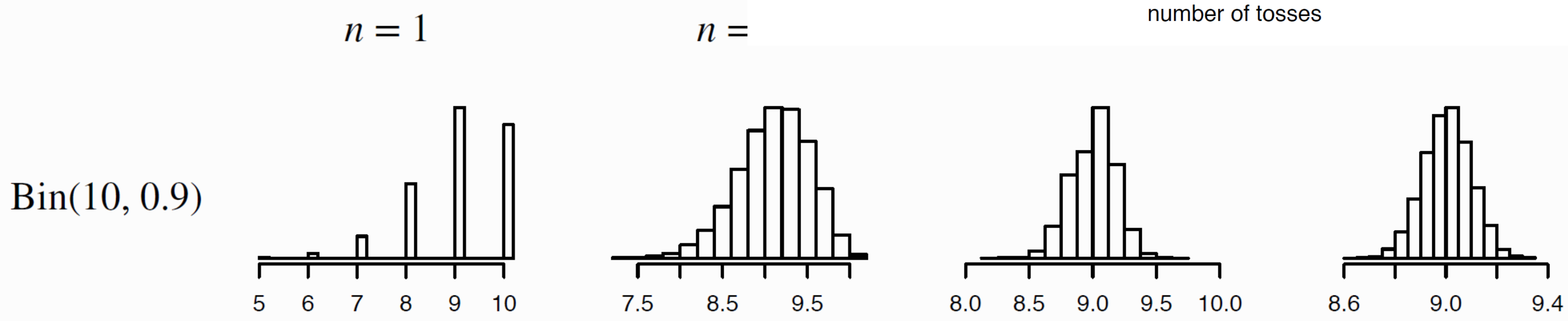
Low resolution 10 Downing St.



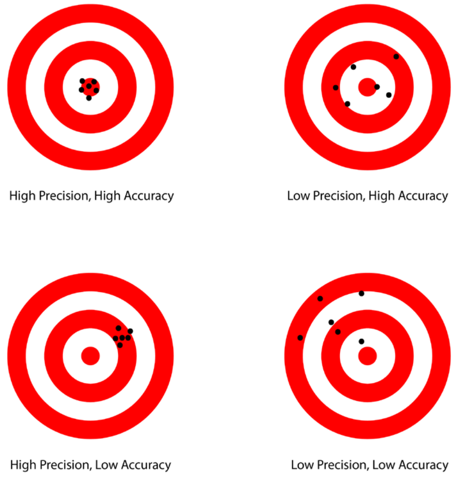
Take-home Messages/W9-3



- *Law of Large Numbers:*
  - Concept of i.i.d. random variables
  - Mean and Variance



- *Central Limit Theorem*
- *Estimation Theory:*
  - Examples of estimators, MLE (Maximum Likelihood Estimator)
  - Example: Positron Emission Tomography ↔ different time-of-arrival estimators
- *Precision, Accuracy, Resolution*



Appendix A: Gamma Distribution – Gamma Function

- While the Exponential distribution represents the wait time before the first success under the conditions of memorylessness, the gamma distribution represents the *total waiting time for multiple successes* (hence it is the sum of multiple exponential distributions).

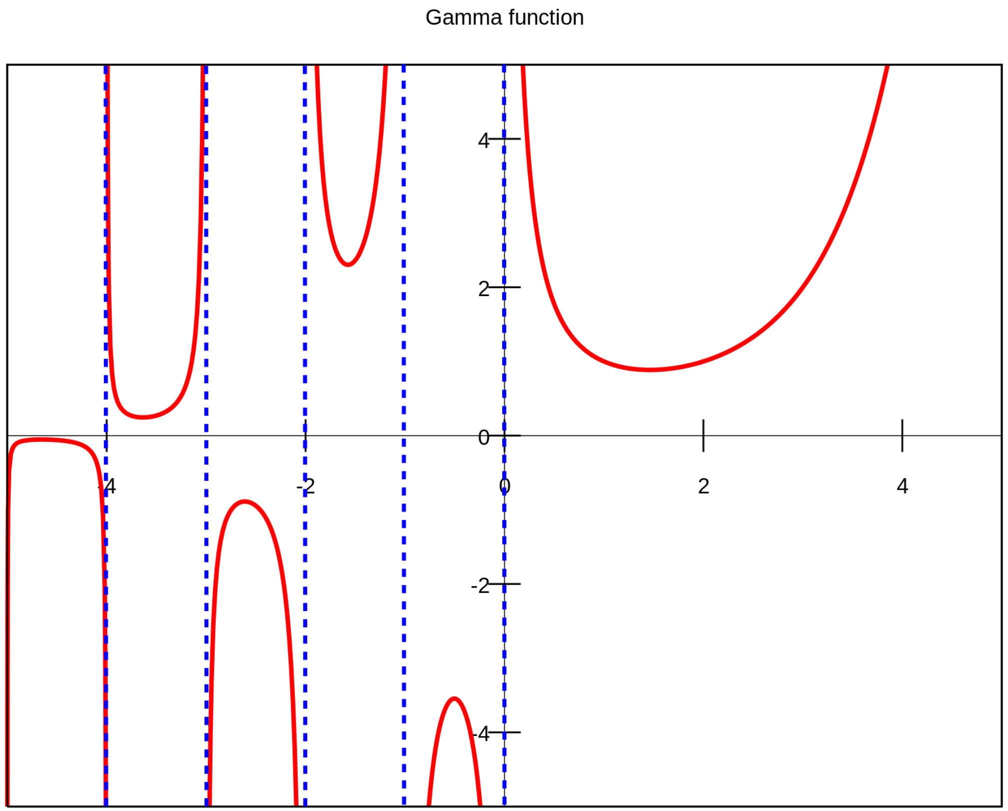
- We first define the gamma function as:

$$\Gamma(a) = \int_0^\infty x^a e^{-x} \frac{dx}{x}, \qquad a > 0$$

- The gamma function has the following properties:

$$\Gamma(a+1) = a\,\Gamma(a)$$

$$\Gamma(n) = (n-1)!$$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

Appendix A: Gamma Distribution (contd.)

- Then, we say that X has a gamma distribution (we will write $X \sim \mathrm{Gamma}(a,1)$) if:

$$\text{PDF:} \qquad f_X(x) = \frac{1}{\Gamma(a)} x^a e^{-x} \frac{1}{x}, \qquad x > 0$$

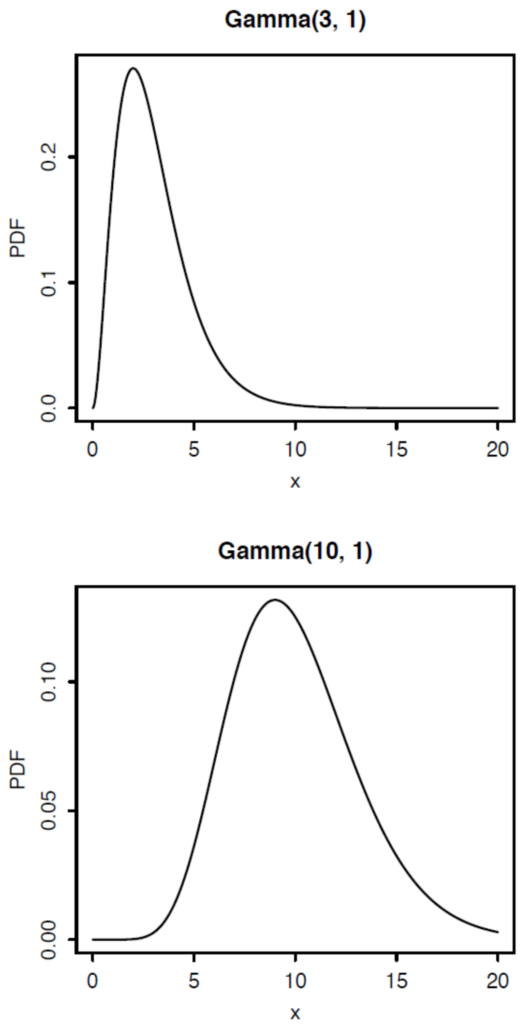
- From the gamma distribution of $X \sim \mathrm{Gamma}(a,1)$, we get, for $\lambda > 0$, the more general $Y = X/\lambda \sim \mathrm{Gamma}(a,\lambda)$:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\Gamma(a)} (\lambda y)^a e^{-\lambda y} \frac{1}{\lambda y} \lambda$$

hence

$$\text{PDF:} \qquad f_Y(y) = \frac{1}{\Gamma(a)} (\lambda y)^a e^{-\lambda y} \frac{1}{y}, \qquad y > 0$$

Gamma$(a,1)$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

Appendix A: Gamma Distribution (contd.)

- From the PDF of the gamma distribution just obtained $Y \sim \text{Gamma}(a, \lambda)$, it can be shown that the Gamma is nothing else but the distribution obtained by summing up a independent exponential distributions. In fact, for $a = 1$:

$$\text{PDF:} \qquad f_Y(y) = \frac{1}{\Gamma(a)} (\lambda y)^a e^{-\lambda y} \frac{1}{y}, \qquad y > 0$$

reduces to

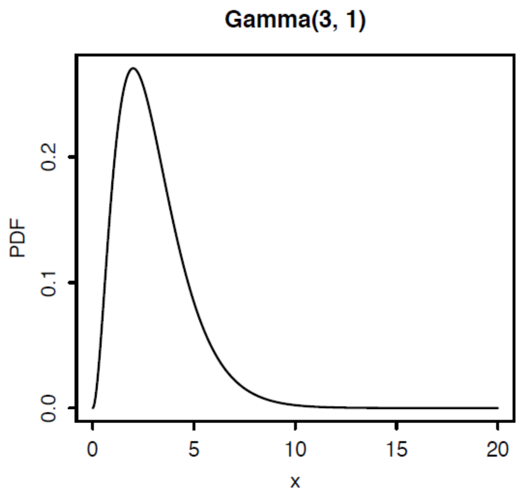
$$\text{PDF:} \qquad f_Y(y) = \lambda y \, e^{-\lambda y} \frac{1}{y} = \lambda \, e^{-\lambda y}, \qquad y > 0$$

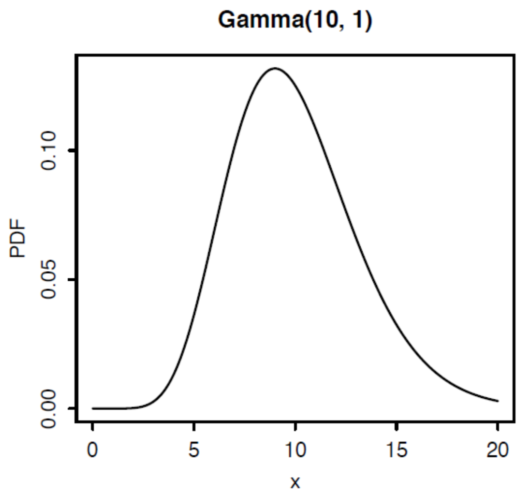
which is the exponential distribution.

- Follows that, let $X_1, X_2, \dots, X_n$ be n i.i.d. $\text{Expo}(\lambda)$. Then:

$$Y = X_1 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$$

Gamma$(a, 1)$





J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

Appendix A: Gamma Distribution (contd.)

- For a $X \sim \text{Gamma}(a, 1)$, it follows:

Mean: $$E\{X\} = \int_0^\infty \frac{1}{\Gamma(a)} x^{a+1} e^{-x} \frac{dx}{x} = \frac{\Gamma(a+1)}{\Gamma(a)} = a$$

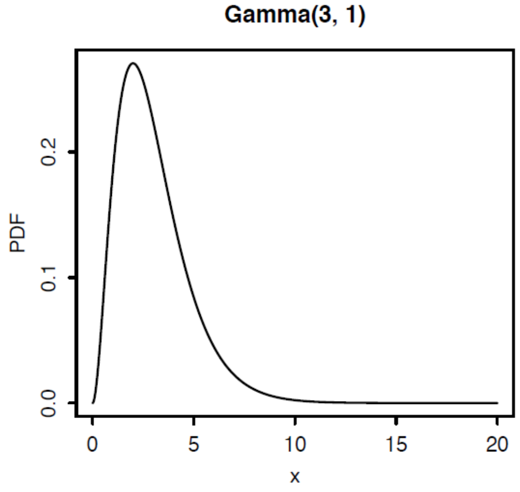
Second Moment: $$E\{X^2\} = \int_0^\infty \frac{1}{\Gamma(a)} x^{a+2} e^{-x} \frac{dx}{x} =$$

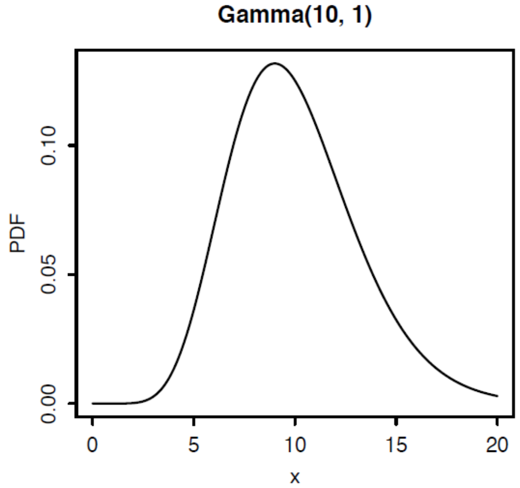
$$= \frac{\Gamma(a+2)}{\Gamma(a)} = a(a+1)$$

Variance: $$Var\{X\} = E\{X^2\} - (E\{X\})^2 =$$

$$= a(a+1) - a^2 = a$$

Gamma$(a, 1)$





J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

Appendix A: Gamma Distribution (contd.)

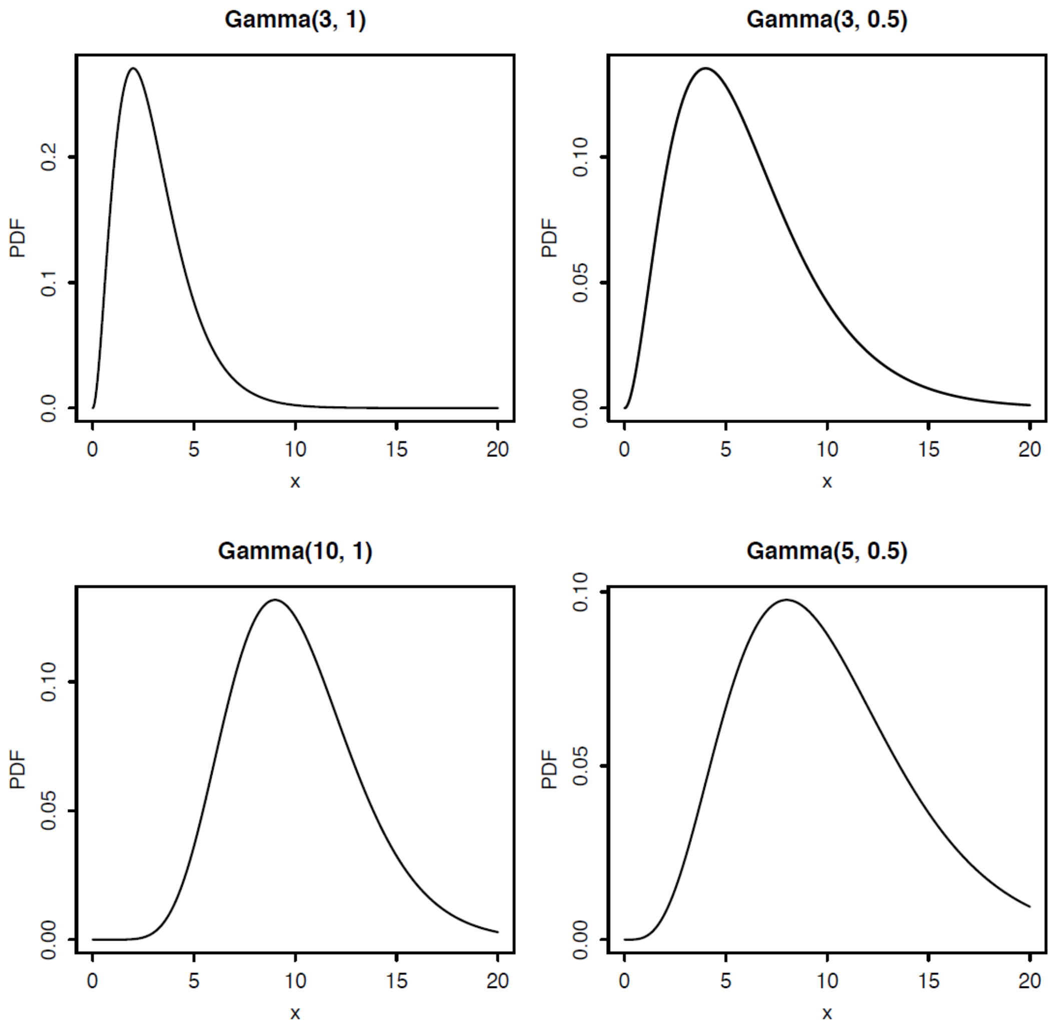
Gamma(a,λ)

- For the more general gamma distribution $Y = X/\lambda \sim \text{Gamma}(a, \lambda)$, by simple transformation, we obtain:

$$\text{Mean:} \qquad E\{Y\} = \frac{1}{\lambda} E\{X\} = \frac{a}{\lambda}$$

$$\text{Second Moment:} \qquad E\{Y^2\} = \frac{1}{\lambda^2} E\{X\} = \frac{a\,(a+1)}{\lambda^2}$$

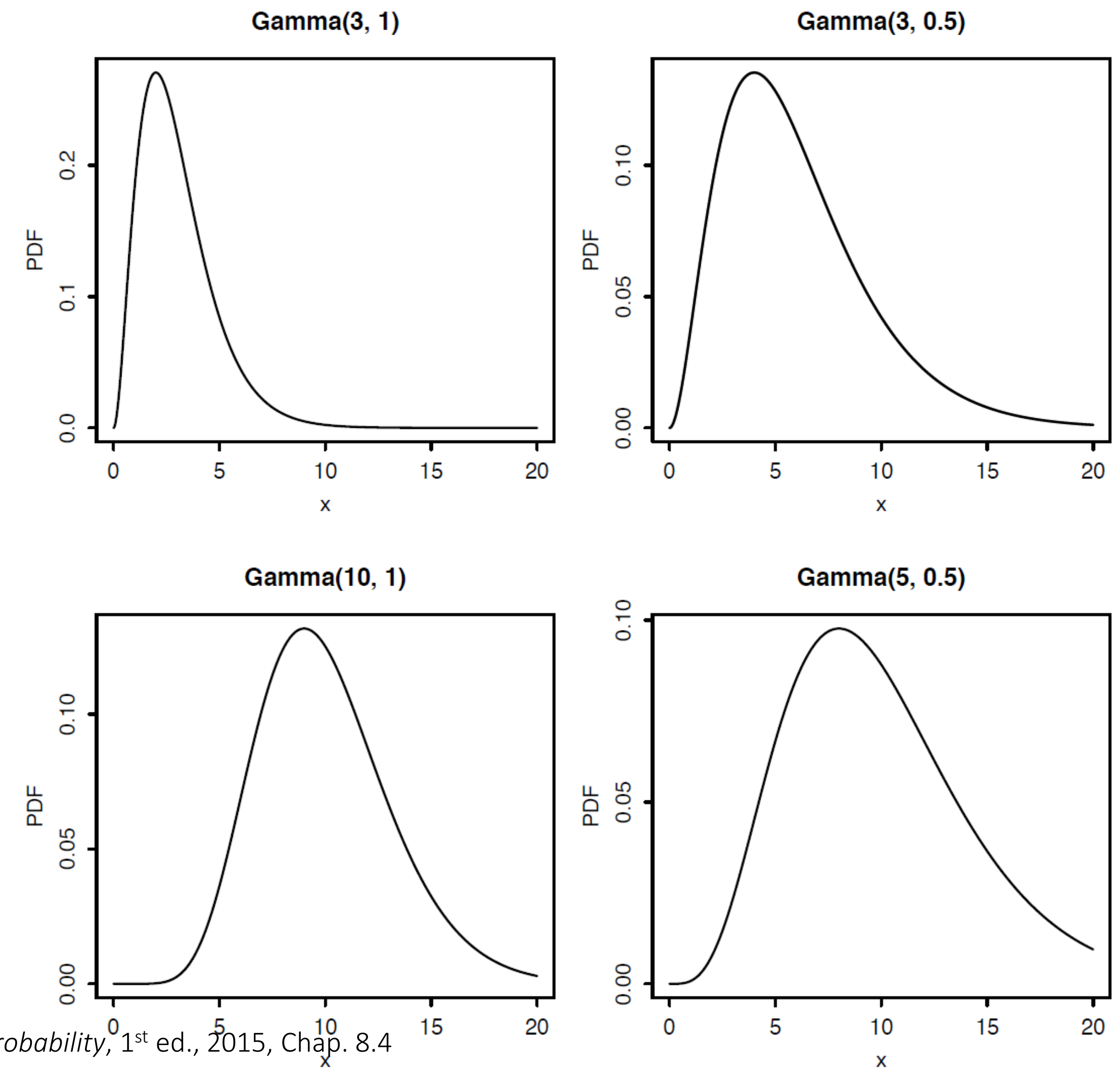
$$\text{Variance:} \qquad Var\{Y\} = \frac{1}{\lambda^2} Var\{X\} = \frac{a}{\lambda^2}$$



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

-> calculate mean/variance for some examples

Appendix A: Gamma Distribution (contd.)



Gamma(a,λ)

-> calculate mean/variance for some examples

Mean: $\frac{a}{\lambda}$

Variance: $\frac{a}{\lambda^2}$

📖 J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 8.4

Appendix B: Poisson Process

- Definition: a sequence of arrivals in continuous time with rate λ is a (1D) Poisson process **with rate** λ if the following two conditions hold:

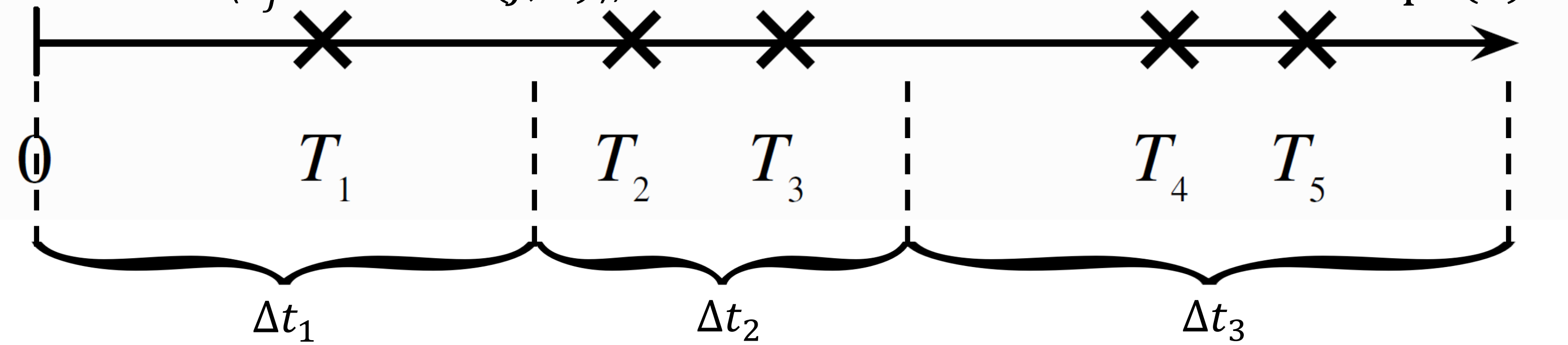
  1) The number of arrivals that occur in an interval of length t is a $\mathrm{Pois}(\lambda t)$ RV.

  2) The numbers of arrivals that occur in disjoint intervals – e.g. (0,10), [10,12) and [15,∞) – are independent of each other.

- If T_j is the time of the j-th arrival, $N(t)$ is the number of events up to the time t, follows:

$$P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

so T_1 has an Exponential distribution ($T_1 \sim \mathrm{Expo}(\lambda)$), hence T_j, being the sum of j i.i.d. exponentials, is a Gamma distribution ($T_j \sim \mathrm{Gamma}(j, \lambda)$), and the interarrival times are i.i.d. $\mathrm{Expo}(\lambda)$ RVs.



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 5.6, 13

Appendix B: Poisson Process

- NB: i.i.d. = independent and identically distributed Random Variables, have the same PDF and are all mutually independent

- *"Confirmation" that the Exponential distribution is closely connected to the Poisson distribution!*
- Examples of Poisson processes:
  - 1D: cars passing by a highway checkpoint;
  - 2D: flowers in a meadow;
  - 3D: stars in a region of the galaxy."

Dark Counts and "real" detections in a SPAD sensor

J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 13

Appendix B: Poisson Process

- Timeline: $(0, +\infty)$ but it could also be $(-\infty, +\infty)$

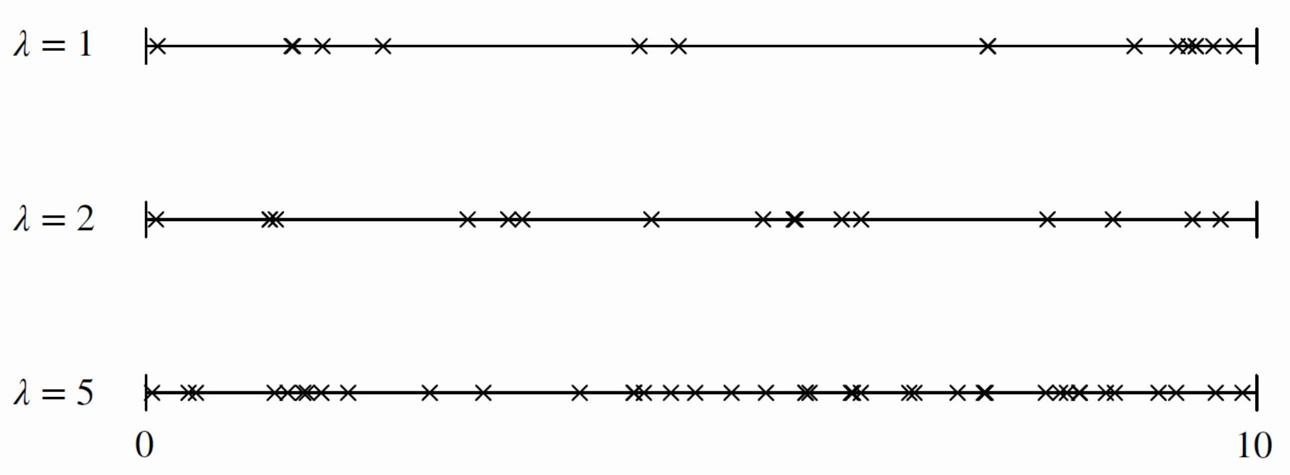
- To generate n arrivals from a Poisson process with rate λ:

  1. Generate n i.i.d. $\mathrm{Expo}(\lambda)$ RVs: $X_1, X_2, \dots, X_n$

  2. For $j = 1, 2, \dots, n$ set $T_j = X_1 + \cdots + X_j$

- Then we can take the $T_1, \dots, T_n$ to be the arrival times.

Simulate Poisson Processes in 1D



Note: interarrival times are i.i.d., but the arrivals are not evenly spaced -> there is a lot of variability in the interarrival times, which produces *Poisson clumping*

📖 J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 13

Appendix B: Poisson Process

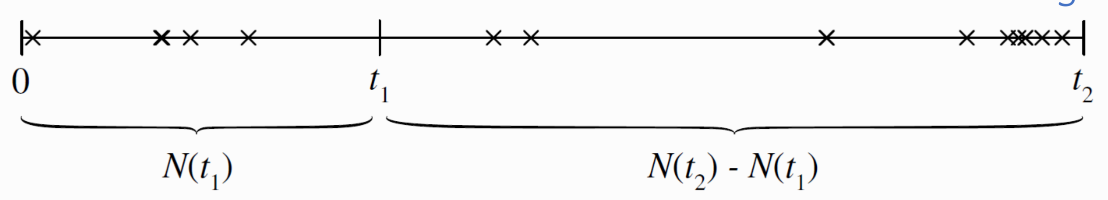
- A Poisson Process has the following three properties:

1. *Conditioning*: let $\{N(t), t > 0\}$ be a Poisson Process with rate λ and $t_2 > t_1$. Then the conditional distribution stands:

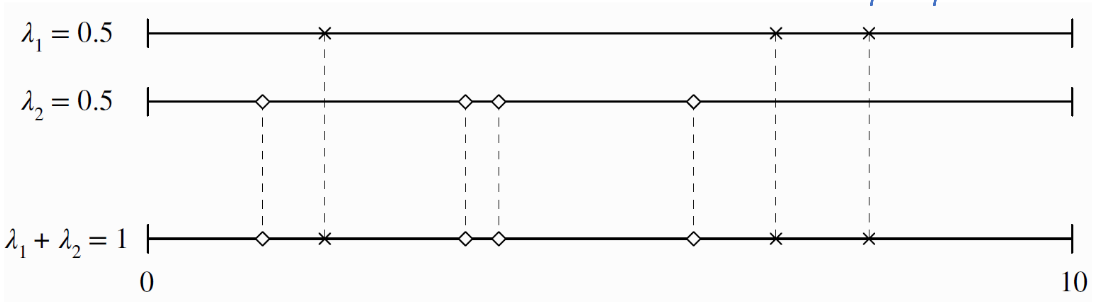
$$N(t_1)\,|\,N(t_2) = n \sim \text{Bin}\left(n, \frac{t_1}{t_2}\right)$$

2. *Superposition*: let $\{N_1(t), t > 0\}$ and $\{N_2(t), t > 0\}$ be two independent Poisson Processes with rates λ_1 and λ_2. Then the combined process $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

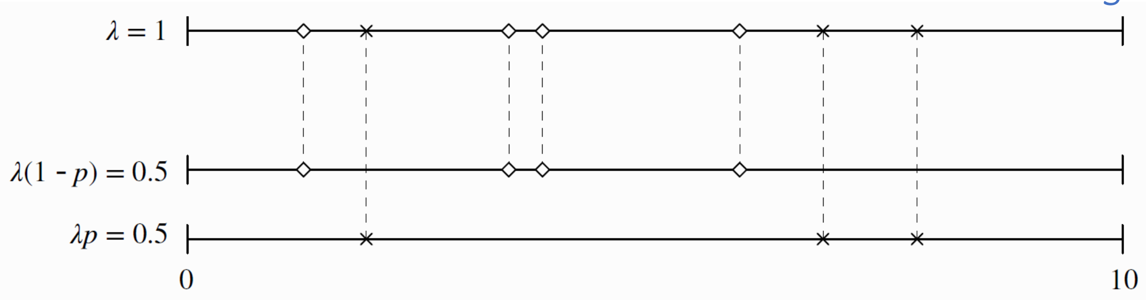
Conditioning



Superposition



Thinning



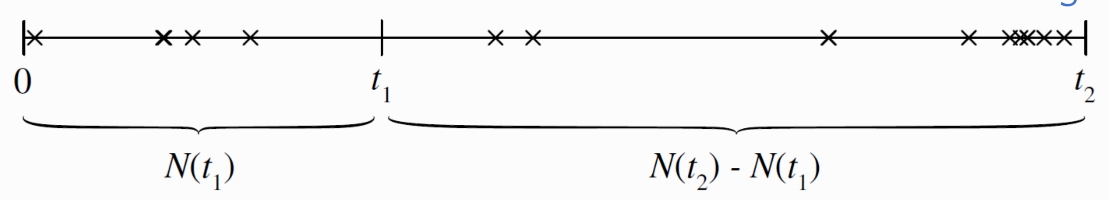
J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 13

Appendix B: Poisson Process

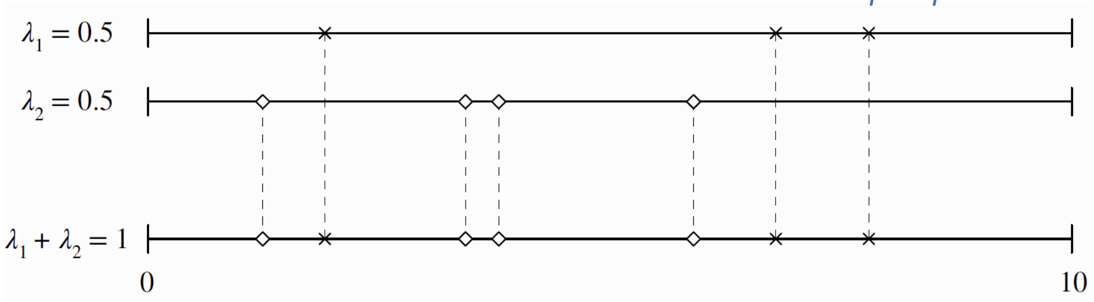
- A Poisson Process has the following three properties:

  3. *Thinning*: let $\{N(t), t > 0\}$ be a Poisson Process with rate λ, and classify each event at the arrival as either type-1 events (with probability p) or type-2 events (with probability $1 - p$), independently. Then the type-1 events form a Poisson process with rate λp, the type-2 events form a Poisson process with rate $\lambda(1 - p)$ and they are independent.

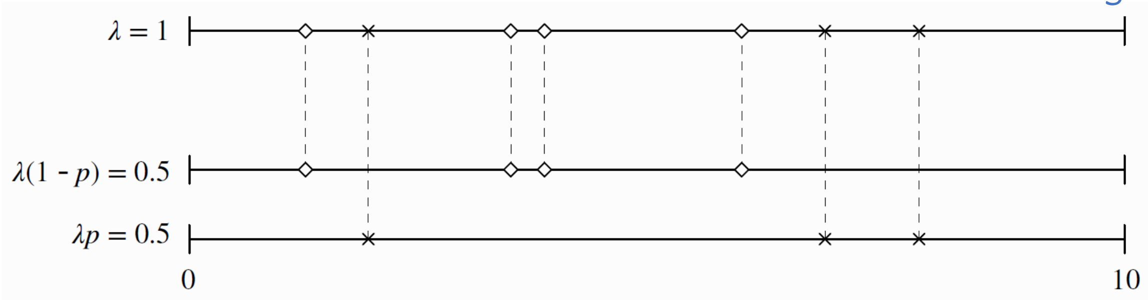
Conditioning



Superposition



Thinning



J.K. Blitzstein, J. Hwang, *Introduction to Probability*, 1st ed., 2015, Chap. 13